University Physics 2

Alvin Lin

January 2018 - May 2018

Electric Flux

Light, magnetism, heat, and water can all have flux. Flux is defined as the flow of some field through a surface.

$$\Phi_E = \vec{E} \cdot \vec{A}$$
$$\vec{A} = A\vec{n}$$

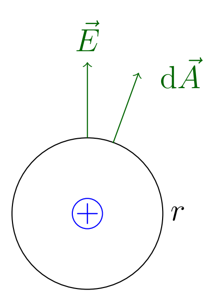
$\vec{A}$ is an area centered on the surface, with the magnitude equal to the area and the direction pointing normal to the surface. For a figure with multiple surfaces:

$$\Phi_E = \sum \vec{E} \cdot \Delta\vec{A}$$

Gauss's Law states that the electric flux through a closed surface is proportional to the charge inside the surface (denoted Q_{enc}).

$$\Phi_E = \frac{Q_{enc}}{\epsilon_\circ} \quad K = \frac{1}{4\pi\epsilon_\circ}$$
$$\Phi_E = \oint \vec{E} \cdot \, \mathrm{d}\vec{A} = \frac{Q_{enc}}{\epsilon_\circ}$$

Gauss's Law for a Point Charge



Use Gauss's Law to find $\vec{E}$ at r given a point charge surrounded by a spherical shell of radius r:

$$
\begin{aligned}
\oint \vec{E} \cdot \, \mathrm{d}\vec{A} &= \frac{Q_{enc}}{\epsilon_\circ} \\
&= \oint |\vec{E}||\, \mathrm{d}\vec{A}| \cos\theta \\
\theta &= 0 \text{ via spherical symmetry} \\
&= \oint |\vec{E}||\, \mathrm{d}\vec{A}| \\
&= |\vec{E}| \oint |\, \mathrm{d}\vec{A}| \text{ since } E \text{ is constant over } r \\
&= |\vec{E}| 4\pi r^2 \\
|\vec{E}| 4\pi r^2 &= \frac{Q}{\epsilon_\circ} \\
|\vec{E}| &= \frac{1}{4\pi r^2} \frac{Q}{\epsilon_\circ} \text{ (Coulomb's Law)}
\end{aligned}
$$

Solving Gauss's Law

1. Draw a picture with axes, distances, coordinates, $\mathrm{d}q$, $\vec{E}$ fields, $\mathrm{d}\vec{A}$.

2. Decide how to solve the problem. For 3D symmetric charge distribution, use Gauss's Law. For 2D rings or lines, use the integral form of Coulomb's Law.

3. Pick a Gaussian surface. For objects with spherical symmetry, use a shell. For objects with cylindrical symmetry, use a hollow cylinder. For objects with planar symmetry, use a box or cylinder.

4. Write Gauss's Law:
$$\oint \vec{E} \cdot \, \mathrm{d}\vec{A} = \frac{Q_{enc}}{\epsilon_\circ}$$

5. Use symmetry to solve the left hand side. Always pick a Gaussian surface so that $\vec{E}$ is parallel to $\mathrm{d}\vec{A}$ so that $\vec{E} \cdot \, \mathrm{d}\vec{A} = |\vec{E}||\, \mathrm{d}\vec{A}|$.

6. Also argue via symmetry that $|\vec{E}|$ is independent of integrating variables so that $\int |\vec{E}||\, \mathrm{d}\vec{A}| = |\vec{E}| \oint \, \mathrm{d}\vec{A}$.

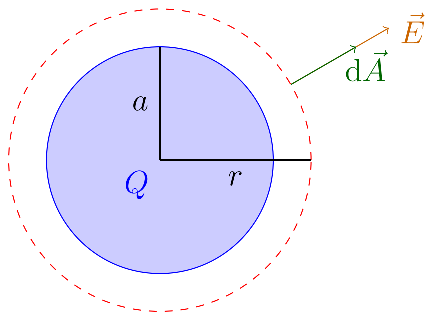
7. Solve the integral for the area. $|\vec{E}| \oint \mathrm{d}\vec{A} = |\vec{E}|A$. For a shell, $A = 4\pi r^2$. For the side of a cylindrical surface, $A = 2\pi rh$. For the top and bottom of a cylindrical surface, $A = \pi r^2$.

8. Calculate Q_{enc}, this can be easy or hard depending on the problem. Remember that $Q = \rho V$.

9. Solve for $|\vec{E}|$.

Gauss's Law for a Spherical Solid

Case $r > a$:



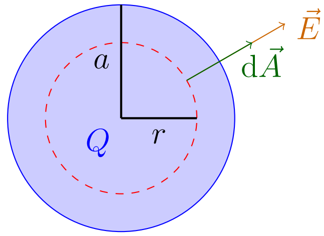
$$\oint \vec{E} \cdot \mathrm{d}\vec{A} = \int |\vec{E}||\mathrm{d}\vec{A}| = |\vec{E}| \int |\mathrm{d}\vec{A}| = |\vec{E}|A = |\vec{E}|4\pi r^2$$

We can make this argument because $\vec{E}$ and $\mathrm{d}\vec{A}$ are both radial and parallel to each other. The electric field is also constant over r by spherical symmetry, so we can bring it outside the integral. Since we are working with a Gaussian surface whose radius r is greater than the radius of the solid a, the enclosed charge is just the total charge.

$$|\vec{E}|4\pi r^2 = \frac{Q_{enc}}{\epsilon_\circ} = \frac{Q}{\epsilon_\circ}$$
$$|\vec{E}| = \frac{1}{4\pi r^2}\frac{Q}{\epsilon_\circ}$$
$$\vec{E} = \frac{1}{4\pi r^2}\frac{Q}{\epsilon_\circ}\hat{r}$$

Note that the formula for the electric field in this case is just like that of a point charge.

Case $r < a$:



$$\oint \vec{E} \cdot \, \mathrm{d}\vec{A} = |\vec{E}| 4\pi r^2$$

This argument can be made for the same reasons as above. The enclosed charge must be expressed in terms of r now. For this problem, we can assume the charge is uniform, which means it can be expressed using the following ratio:

$$\begin{aligned} \rho &= \frac{Q_{total}}{V_{total}} = \frac{Q_{enc}}{V_{enc}} \\ Q_{enc} &= \frac{Q_{total} V_{enc}}{V_{total}} \\ &= \frac{Q}{\frac{4}{3}\pi a^3} \frac{4}{3}\pi r^3 \\ &= \frac{Q r^3}{a^3} \end{aligned}$$

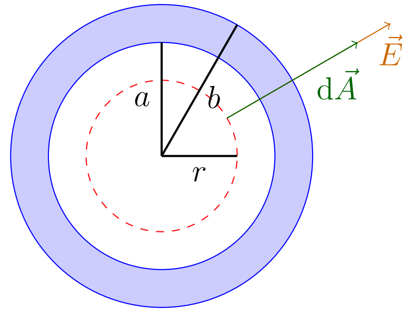
Now we can solve for $|\vec{E}|$:

$$\begin{aligned} |\vec{E}| 4\pi r^2 &= \frac{Q_{enc}}{\epsilon_\circ} = \frac{Q r^3}{\epsilon_\circ a^3} \\ |\vec{E}| &= \frac{1}{4\pi r^2} \frac{Q r^3}{\epsilon_\circ a^3} = \frac{1}{4\pi} \frac{Q r}{\epsilon_\circ a^3} \end{aligned}$$

From these two results, we can see that the electric field increases linearly inside the solid up to a, and then decreases proportionally to $\frac{1}{r^2}$.

Gauss's Law for a Spherical Hollow Shell

Case $r < a$:



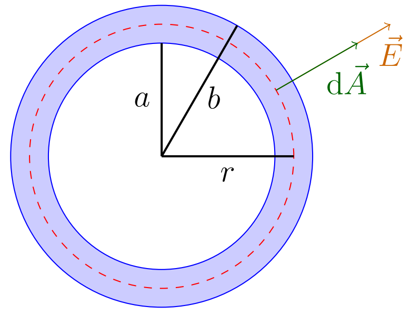
$$\oint \vec{E} \cdot \, \mathrm{d}\vec{A} = |\vec{E}| 4\pi r^2$$

The same argument as above can be applied here. Since the Gaussian surface is inside the solid, it encloses no charge, meaning $Q_{enc} = 0$.

$$|\vec{E}| 4\pi r^2 = \frac{Q_{enc}}{\epsilon_\circ} = 0$$
$$|\vec{E}| = 0$$
$$\vec{E} = \vec{0}$$

The electric field anywhere inside the solid is zero.

Case $a < r < b$:



$$\oint \vec{E} \cdot \, \mathrm{d}\vec{A} = |\vec{E}| 4\pi r^2$$

The same argument as above can be applied here. The enclosed charge must be expressed in terms of r now.

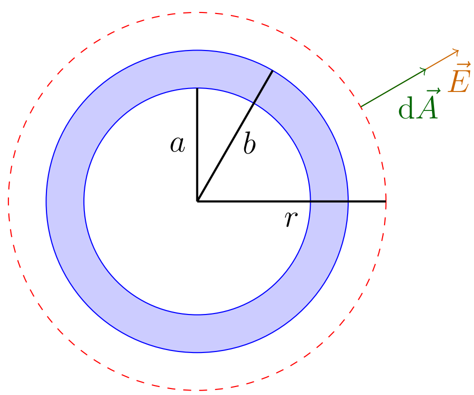
$$Q_{enc} = \int \rho \, \mathrm{d}V = \rho V = \rho(\frac{4}{3}\pi r^3 - \frac{4}{3}\pi a^3) = \frac{4}{3}\pi\rho(r^3 - a^3)$$

Now we can plug this back into Gauss's Law:

$$|\vec{E}|4\pi r^2 = \frac{Q_{enc}}{\epsilon_\circ} = \frac{1}{\epsilon_\circ}\frac{4}{3}\pi\rho(r^3 - a^3)$$
$$|\vec{E}| = \frac{1}{4\pi r^2\epsilon_\circ}\frac{4}{3}\pi\rho(r^3 - a^3)$$
$$= \frac{\rho}{3\epsilon_\circ}\frac{r^3 - a^3}{r^2}$$

Note that inside the solid, the charge increases linearly.

Case $r > b$:



$$\oint \vec{E} \cdot \, \mathrm{d}\vec{A} = |\vec{E}|4\pi r^2$$

This same argument as above can be applied here. Like the spherical solid, since the Gaussian surface is larger than the solid, the enclosed charge is just the total charge.

$$|\vec{E}|4\pi r^2 = \frac{Q_{enc}}{\epsilon_\circ} = \frac{Q}{\epsilon_\circ}$$
$$|\vec{E}| = \frac{1}{4\pi r^2}\frac{Q}{\epsilon_\circ}$$
$$\vec{E} = \frac{1}{4\pi r^2}\frac{Q}{\epsilon_\circ}\hat{r}$$

Notice that this formula is again very much like that of a point charge. For this spherical hollow, it can also be written as:

$$|\vec{E}| = \frac{\rho}{3\epsilon_\circ}\frac{b^3 - a^3}{r^2}$$

Important Relations

$$\begin{aligned}
\lambda &= \frac{Q}{L} \\
\sigma &= \frac{Q}{A} \\
\rho &= \frac{Q}{V} = \frac{Q}{AL} = \frac{\lambda}{A} = \frac{\sigma}{L} \\
Q_{total} &= \int \rho \, \mathrm{d}V
\end{aligned}$$

Rules for Conductors

The Gauss Law applications above apply for insulting solids. For conductors, we must observe a few rules:

- Conductors (metals) allow for the flow of charge.
- $Q_{enc} = 0$ inside the conductor since all charges are on the surface. Therefore the electric field is also zero inside a conductor.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech