University Physics 2

Alvin Lin

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Electric Fields

We know positive charges feel repulsive forces from other positive charges. If we take a test charge and place it near an object of interest, how it would experience a force is the direction of the electric field, also known as the E-field. We define it as:

$$\vec{E} = \frac{\vec{F_\circ}}{q_\circ}$$

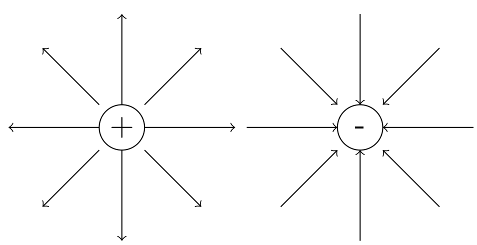
where $\vec{F_\circ}$ is the force on the test charge, $q_\circ$ is the test charge, and $\vec{E}$ is the electric field from some charge distribution. Usually, we will represent this in the form:

$$\vec{F} = q\vec{E}$$

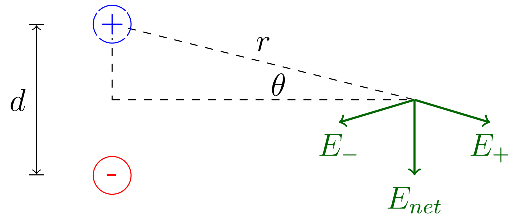
This allows us to calculate the force $\vec{F}$ for some charge q exerted by an external electric field $\vec{E}$. For a point charge, we calculate the field with:

$$\overrightarrow{E_{pc}} = \frac{Kq\hat{r}}{r^2}$$

This is the same as Coulomb's Law but is measured in Newtons per Coulomb. This is due to the fact that there is no second charge from the observation charge. The field lines point away from positive charges and towards negative charges.



Electric Dipoles: Off-Axis Field



The x-components of the fields cancel by symmetry, so we are only concerned with the y-components.

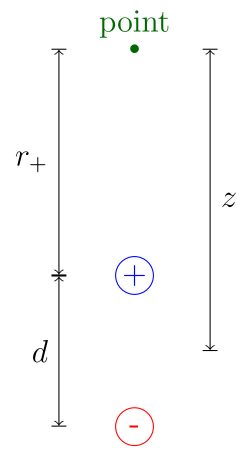
$$|\overrightarrow{E_+}| = \frac{kq}{r^2}$$
$$E_{+,y} = |E_+|\sin\theta = \frac{kq\sin\theta}{r^2}$$

In the case where $\frac{d}{x} << 1$:

$$\left(\frac{d}{x}\right)^2 \approx 0 \quad \sin\theta = \frac{\frac{d}{2}}{r} \quad r = \sqrt{x^2 + \left(\frac{d}{2}\right)^2}$$

$$\begin{aligned}
E_{+,y} &= \frac{kq\sin\theta}{r^2} = \frac{kq}{r^2}\frac{\frac{d}{2}}{r} = \frac{1}{2}\frac{kqd}{r^3} \\
&= \frac{1}{2}\frac{kqd}{\left(\sqrt{x^2 + \frac{d^2}{4}}\right)^3} \\
&= \frac{1}{2}kqd\left[x^2 + \frac{d^2}{4}\right]^{-\frac{3}{2}} \\
&= \frac{1}{2}kqdx^{-3}\left[1 + \frac{d^2}{4x^2}\right]^{-\frac{3}{2}} \\
&\approx \frac{1}{2}kqdx^{-3}\left(1 + 0\right)^{-\frac{3}{2}} \\
&\approx \frac{kqd}{2x^3} \\
|\overrightarrow{E_{net}}| &= 2E_{+,y} = \frac{kqd}{x^3}
\end{aligned}$$

Electric Dipoles: On-Axis Field

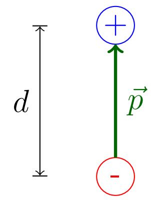


In the case where $d << x$, we can use the binomial expansion:

$$
\begin{aligned}
(1+\epsilon)^n &\approx 1 + n\epsilon \\
|\overrightarrow{E_+}| &= \frac{kq}{r_+^2} = \frac{kq}{(z-\frac{d}{2})^2} \\
&= kqz^{-2}(1-\frac{d}{2z})^{-2} \\
&\approx kqz^{-2}\left(1+(-2)(\frac{-d}{2n})\right) \\
&\approx kqz^{-2}(1+\frac{d}{z}) \\
|\overrightarrow{E_-}| &= \frac{kq}{(r_+ + d)^2} = \frac{kq}{(z+\frac{d}{2})^2} \\
&= kqz^{-2}(1+\frac{d}{2z})^{-2} \\
&\approx kqz^{-2}(1-\frac{d}{z}) \\
|\overrightarrow{E_{net}}| &= E_+ - E_- \\
&= (\frac{kq}{z^2}+\frac{kqd}{z^3}) - (\frac{kq}{z^2}-\frac{kqd}{z^3}) \\
&= \frac{2kqd}{z^3}
\end{aligned}
$$

Note that for a point charge, $E \sim \frac{1}{r^2}$, and for a dipole $E \sim \frac{1}{r^3}$.

Dipole Moment



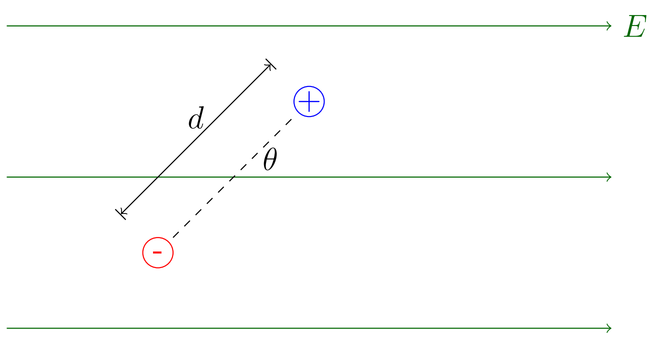
In this diagram, $\vec{p}$ is the dipole moment. It points from negative to positive and has a magnitude of qd. For the on-axis field:

$$\vec{E} = \frac{2k\vec{p}}{z^3} \quad \vec{p} = qd\hat{j}$$

For the off-axis field:

$$\vec{E} = \frac{-k\vec{p}}{z^3} \quad \vec{p} = qd\hat{j}$$

Dipoles in Electric Fields



$$\vec{F} = q\vec{E} + (-q\vec{E}) = 0$$
$$|\vec{\tau}| = dq\vec{E}\sin\theta = \vec{p}\vec{E}\sin\theta$$
$$\vec{\tau} = \vec{p} \times \vec{E}$$

Electric Fields of Solid Objects

Strategy for calculating the electric field of a solid object:

1. Draw a diagram: draw the axes, place the charge, draw $\mathrm{d}q$, and any distances $(r, x, \theta, \ldots)$.

2. Calculate the charge density λ. If it is uniform, then $\lambda = \frac{Q}{L}$. Otherwise, it is a function $\lambda(r)$ and we need to integrate.

3. Figure out $\mathrm{d}q$. $\mathrm{d}q = \lambda\ \mathrm{d}s$. For a line $\mathrm{d}s = \mathrm{d}x$. For an arc, $\mathrm{d}s = r\ \mathrm{d}\theta$.

4. Find $|\ \mathrm{d}\vec{E}|$
$$|\ \mathrm{d}\vec{E}| = \frac{k\ \mathrm{d}q}{r^2}$$

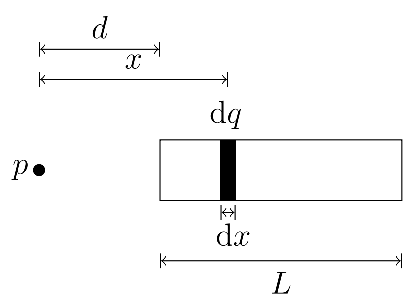
5. Find the components.
$$\mathrm{d}E_x = |\ \mathrm{d}\vec{E}| \cos\theta$$
$$\mathrm{d}E_y = |\ \mathrm{d}\vec{E}| \sin\theta$$
Note that this depends on whether or not θ is from the horizontal.

6. Get the limits of integration from the diagram.

7. Check for symmetries to see if any integrals are zero.

8. Integrate:
$$E_x = \int\ \mathrm{d}E_x$$

Example



$$r = x \quad \lambda = \frac{Q}{L} \qquad \mathrm{d}q = \lambda\ \mathrm{d}s = \frac{Q}{L}\ \mathrm{d}x$$
$$\begin{aligned} |\vec{E}| &= \int_d^{d+L} \frac{k}{r^2}\ \mathrm{d}q = \int_d^{d+L} \frac{kQ}{Lx^2}\ \mathrm{d}x \\ &= \frac{kQ}{L}\left[\frac{-1}{x}\right]_d^{d+L} \\ &= \frac{kQ}{d(d+L)} \end{aligned}$$

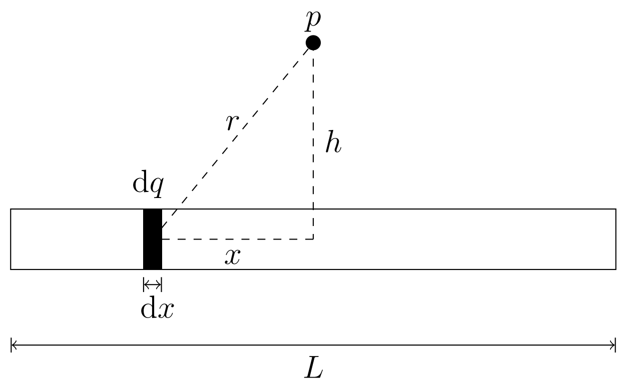
Let's consider the field where the point is very far away from the rod.

$$d >> L \Rightarrow \frac{L}{d} << 1 \Rightarrow \frac{L}{d} \sim 0$$

In this case, where the point is very far away, the field of the rod looks like the field of a point charge:

$$|\vec{E}| = \frac{kQ}{(d(d+L))} = \frac{kQ}{d^2(1+\frac{L}{d})} \sim \frac{kQ}{d^2}$$

Example



By symmetry, we can see that the field in the x-direction cancels out, but we will solve it for completeness. Our known values are as follows:

$$|\vec{E}| = \frac{kQ}{r^2} \quad |\,\mathrm{d}\vec{E}| = \frac{k\ \mathrm{d}q}{r^2}$$

$$r^2 = x^2 + h^2 \quad \lambda = \frac{Q}{L}$$

$$\mathrm{d}q = \lambda\ \mathrm{d}s = \frac{Q}{L}\ \mathrm{d}x$$

$$\cos\theta = \frac{x}{h}$$

$$\sin\theta = \frac{h}{r} = \frac{h}{\sqrt{x^2+h^2}}$$

Since the point is on a different axis than the rod, we have to separate the electric field into components.

$$
\begin{aligned}
\mathrm{d}E_x &= |\,\mathrm{d}\vec{E}|\cos\theta \\
&= \frac{kQ}{Lr^2}\frac{x}{r}\,\mathrm{d}x \\
&= \frac{kQ}{L}\frac{x}{(x^2+h^2)^{\frac{3}{2}}}\,\mathrm{d}x \\
E_x &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{kQ}{L}\frac{x}{(x^2+h^2)^{\frac{3}{2}}}\,\mathrm{d}x = 0 \\
\mathrm{d}E_y &= |\,\mathrm{d}\vec{E}|\sin\theta \\
&= \frac{kQ}{Lr^2}\frac{h}{r}\,\mathrm{d}x \\
&= \frac{kQh}{L}\frac{1}{(h^2+x^2)^{\frac{3}{2}}}\,\mathrm{d}x \\
E_y &= \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{kQh}{L}\frac{1}{(x^2+h^2)^{\frac{3}{2}}}\,\mathrm{d}x \\
&= \frac{kQ}{h\sqrt{h^2+\frac{L^2}{4}}}
\end{aligned}
$$

Since the field in the x-direction cancels out:

$$|\vec{E}| = E_y = \frac{kQ}{h\sqrt{h^2+\frac{L^2}{4}}}\hat{j}$$

Let's consider two limits: suppose the point is very far away from the rod:

$$h >> L \Rightarrow \frac{L}{h} << 1 \Rightarrow \frac{L}{h} \sim 0$$

Like before, the field of the rod looks like that of a point charge:

$$|\vec{E}| = \frac{kQ}{h\sqrt{h^2+\frac{L^2}{4}}} = \frac{kQ}{h^2(1+\frac{L^2}{4h^2})} \sim \frac{kQ}{h^2}$$

Let's consider now a very long rod (or having the point inside the rod):

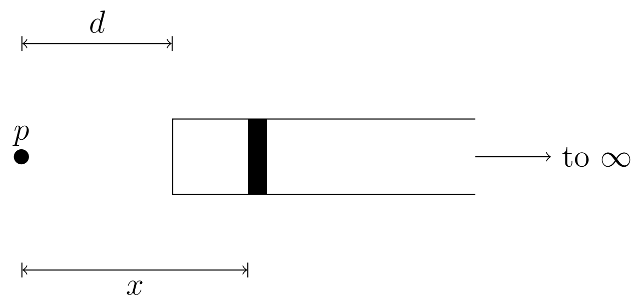
$$L >> h \Rightarrow \frac{h}{L} \sim 0$$

This gives us a very interesting result:

$$|\vec{E}| = \frac{kQ}{h\sqrt{h^2 + \frac{L^2}{4}}} = \frac{kQ}{hL\sqrt{\frac{h^2}{L^2} + \frac{1}{4}}} = \frac{kQ}{hL\sqrt{\frac{1}{4}}} = \frac{2kQ}{hL} = \frac{2k\lambda}{h}$$

This is the electric field of a rod of infinite length.

Example



In this example, there is an infinite line of charge along the x-axis with a non-uniform charge density of $\lambda = \beta x^{-3}$. The units of β are given in Coulomb $\cdot$ meters squared.

$$\begin{aligned}
\mathrm{d}q &= \lambda \ \mathrm{d}x \\
&= \frac{\beta}{x^3} \ \mathrm{d}x \\
Q &= \int_d^\infty \frac{\beta}{x^3} \ \mathrm{d}x \\
&= \left[\beta \frac{-1}{2} x^{-2}\right]_d^\infty \\
&= -\frac{\beta}{2}\left(\frac{1}{\infty^2} - \frac{1}{d^2}\right) \\
&= \frac{\beta}{2d^2}
\end{aligned}$$

$$
\begin{aligned}
|\,\mathrm{d}\vec{E}| &= \frac{k\ \mathrm{d}q}{r^2} \\
&= \frac{k\beta}{x^5}\ \mathrm{d}x \\
|\vec{E}| &= \int_d^\infty \frac{k\beta}{x^5}\ \mathrm{d}x \\
&= k\beta\left[\frac{-1}{4}x^{-4}\right]_d^\infty \\
&= -\frac{k\beta}{4}\left(\frac{1}{\infty^4} - \frac{1}{d^4}\right) \\
&= \frac{k\beta}{4d^4}
\end{aligned}
$$

The direction of this field is to the left from the point charge p.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech