Advanced Linear Algebra

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Practice Test 1

Question 1

Find the equation of the plane passing through $(0, 1, 0)$ with normal vector $\vec{n} = \langle 3, 2, 1\rangle$.

$$\begin{aligned} \vec{n}\cdot\vec{x} &= \vec{n}\cdot\vec{p} \\ \langle 3,2,1\rangle\cdot\vec{x} &= \langle 3,2,1\rangle\cdot\langle 0,1,0\rangle \\ 3x+2y+z &= 0+2+0 \\ 3x+2y+z &= 2 \end{aligned}$$

Question 2

Consider the plane $x + y - z = 1$.

(a) What is the normal vector $\vec{n}$ to the plane?

$$\vec{n} = \langle 1, 1, -1\rangle$$

(b) Consider the point on the plane $A(1, 0, 0)$ and the point off of the plane $B(1, 0, 2)$. Find the projection of the vector $\overrightarrow{AB}$ onto $\vec{n}$.

$$\begin{aligned} \overrightarrow{AB} &= \langle 1-1, 0-0, 2-0\rangle = \langle 0,0,2\rangle \\ \text{proj}_{\vec{n}}\overrightarrow{AB} &= \frac{\overrightarrow{AB}\cdot\vec{n}}{\vec{n}\cdot\vec{n}}\vec{n} \\ &= \frac{0+0-2}{1+1+1}\langle 1,1,-1\rangle \\ &= \langle -\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\rangle \end{aligned}$$

(c) What is the distance from point B to the plane?

$$\begin{aligned}\|\langle -\frac{2}{3}, -\frac{2}{3}, \frac{2}{3}\rangle\| &= \sqrt{\frac{12}{9}} \\ &= \frac{2\sqrt{3}}{3}\end{aligned}$$

Question 3

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{bmatrix}$$

(a) Find the reduced row echelon form $R = rref(A)$. What is the rank of A?

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 \\ 1 & 2 & 1 & 3 \end{bmatrix} (R_1 - R_3 \to R_1, R_2 - 2R_3 \to R_2) \\ &= \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} (R_1 \leftrightarrow R_2, -R_2 \to R_2) \end{aligned}$$

$$rank(A) = 2$$

(b) Find a basis for the null space of A.

$$A\vec{x} = 0$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad x_1 + 2x_2 + x_4 = 0 \quad x_3 + 2x_4 = 0$$

$$x_2 = s \quad x_4 = t$$

$$x_1 = -2s - t$$

$$x_3 = -2t$$

$$\vec{x} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

$$null(A) = span \left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right)$$

(c) If the vector $\vec{b}$ is the sum of the four columns of A, write down the complete solution to $A\vec{x} = \vec{b}$.

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

Question 4

Suppose row operations change $A\vec{x} = \vec{b}$ to a row reduced $R\vec{x} = \vec{d}$ and the complete solution is

$$\vec{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

then what is the 3 by 3 reduced row echelon matrix R and what is $\vec{d}$?

$$x = 4 + 2c_1 + 5x_2$$
$$y = c_1$$
$$z = c_2$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 + 2x + 5t \\ s \\ t \end{bmatrix}$$
$$x - 2w - 5y = 4$$
$$R = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\vec{d} = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}$$

Question 5

Suppose A is the matrix

$$\begin{bmatrix} 2 & 1 \\ 6 & 5 \\ 2 & 4 \end{bmatrix}$$

Is the vector $\vec{b} = \begin{bmatrix} 8 \\ 28 \\ 14 \end{bmatrix}$ in the column space of A?

$$A\vec{x} = \vec{b}$$
$$\left[\begin{array}{cc|c} 2 & 1 & 8 \\ 6 & 5 & 28 \\ 2 & 4 & 14 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$
$$x_1 = 3 \quad x_2 = 2$$
$$\vec{b} = 3\begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$$

A solution exists so $\vec{b}$ is in the column space of A.

Question 6

Use row operations to find the inverse of

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 2 & 4 & 0 & 1 & 0 \\ 1 & 3 & -3 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 2 & -1 & 1 & 0 & 0 \\ 0 & -2 & 6 & -2 & 1 & 0 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right] (R_2 - 2R_1 \to R_2, R_3 - R_1 \to R_3)$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & 0 & -2 \\ 0 & 0 & 2 & -1 & 1 & 2 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right] (R_1 - 2R_3 \to R_1, R_2 + 2R_3 \to R_3)$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & 0 & -2 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \\ 0 & 1 & -2 & -1 & 0 & 1 \end{array}\right] (\frac{1}{2}R_2 \to R_2)$$

$$= \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 9 & -\frac{3}{2} & -5 \\ 0 & 1 & 0 & -5 & 1 & 3 \\ 0 & 0 & 1 & -2 & \frac{1}{2} & 1 \end{array}\right] (R_3 + 2R_2 \to R_3, R_1 - 3R_2 \to R_1, R_2 \leftrightarrow R_3)$$

Question 7

Let $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$

(a) Find $\det(AB)$.

$$\det(AB) = \det(A)\det(B)$$

$$\det(A) = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} - 0 + 0 = 0$$

$$\det(AB) = 0$$

(b) Find $\det(B)$.

$$\begin{aligned}\det(B) &= -\begin{vmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 0\end{vmatrix} + \begin{vmatrix}0 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 0\end{vmatrix} - 0 + 0\\ &= 0 + \begin{vmatrix}1 & 1\\ 1 & 1\end{vmatrix} - \begin{vmatrix}0 & 1\\ 1 & 1\end{vmatrix}\\ &= 1\end{aligned}$$

Question 8

Let $A = \begin{bmatrix}0 & -1\\ 4 & 0\end{bmatrix}$.

(a) Find the eigenvalues of A.

$$\begin{aligned}\det(A - \lambda I) &= \begin{vmatrix}-\lambda & -1\\ 4 & -\lambda\end{vmatrix}\\ &= \lambda^2 + 4 = 0\\ \lambda &= \pm 2i\end{aligned}$$

(b) Find the eigenvectors of A.

$$(A - (2i)I)\vec{x} = 0$$

$$\left[\begin{array}{cc|c}-2i & -1 & 0\\ 4 & -2i & 0\end{array}\right] \to \left[\begin{array}{cc|c}2 & -i & 0\\ 0 & 0 & 0\end{array}\right]$$

$$\vec{x} = s\begin{bmatrix}\frac{1}{2}i\\ 1\end{bmatrix}$$

$$(A + (2i)I)\vec{x} = 0$$

$$\left[\begin{array}{cc|c}2i & -1 & 0\\ 4 & 2i & 0\end{array}\right] \to \left[\begin{array}{cc|c}2 & i & 0\\ 0 & 0 & 0\end{array}\right]$$

$$\vec{x} = s\begin{bmatrix}-\frac{1}{2}i\\ 1\end{bmatrix}$$

Question 9

Solve the linear system of coupled ODEs.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -y \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 4x$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$D = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}$$

$$P = \begin{bmatrix} -\frac{1}{2}i & -\frac{1}{2}i \\ 1 & 1 \end{bmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(P^{-1}\vec{x}) = D(P^{-1}\vec{x}) \quad \vec{z} = P^{-1}\vec{x}$$

$$\frac{\mathrm{d}\vec{z}}{\mathrm{d}t} = D\vec{z}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

$$\frac{\mathrm{d}z_1}{\mathrm{d}t} = 2iz_1 \quad z_1 = c_1\mathrm{e}^{2it}$$

$$\frac{\mathrm{d}z_2}{\mathrm{d}t} = -2iz_2 \quad z_2 = c_2\mathrm{e}^{-2it}$$

$$\begin{aligned} \vec{x} &= P\vec{z} \\ &= \begin{bmatrix} -\frac{1}{2}i & -\frac{1}{2}i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1\mathrm{e}^{2it} \\ c_2\mathrm{e}^{-2it} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2}ic_1\mathrm{e}^{2it} - \frac{1}{2}c_2\mathrm{e}^{-2it} \\ c_1\mathrm{e}^{2it} + c_2\mathrm{e}^{-2it} \end{bmatrix} \end{aligned}$$

Question 10

Solve the system of coupled ODEs.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 3x - 4y \quad x(0) = 1$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = 2x - 3y \quad y(0) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$
$$= (PDP^{-1})\begin{bmatrix} x \\ y \end{bmatrix}$$
$$A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix} = PDP^{-1}$$
$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$
$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$\frac{\mathrm{d}(P^{-1}\vec{x})}{\mathrm{d}t} = D(P^{-1}\vec{x}) \quad \vec{z} = P^{-1}\vec{x}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$
$$\frac{\mathrm{d}z_1}{\mathrm{d}t} = z_1 \quad z_1 = c_1\mathrm{e}^t$$
$$\frac{\mathrm{d}z_2}{\mathrm{d}t} = -x_2 \quad x_2 = c_2\mathrm{e}^{-t}$$
$$\vec{x} = P\vec{z}$$
$$= \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} c_1\mathrm{e}^t \\ c_2\mathrm{e}^{-t} \end{bmatrix}$$
$$= \begin{bmatrix} 2c_1\mathrm{e}^t + c_2\mathrm{e}^{-t} \\ c_1\mathrm{e}^t + c_2\mathrm{e}^{-t} \end{bmatrix}$$
$$x(0) = 1 = 2c_1 + c_2$$
$$y(0) = 0 = c_1 + c_2$$
$$c_1 = 1 \quad c_2 = -1$$
$$\vec{x} = \begin{bmatrix} 2\mathrm{e}^t - \mathrm{e}^{-t} \\ \mathrm{e}^t - \mathrm{e}^{-t} \end{bmatrix}$$

Question 11

Consider

$$A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

Find matrices P and D such that $D = P^{-1}AP$.

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix}$$

$$= -\lambda \begin{vmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{vmatrix} - 2 \begin{vmatrix} 1 & 2-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4$$

$$= (\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$$

$$\lambda = 1$$

$$(A - 1I)\vec{v_1} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v_1} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda = 2$$

$$(A - 2I)\vec{v_2} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\vec{v_2} = s \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad P = \begin{bmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech