Advanced Linear Algebra: Homework 4

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Question 1

Orthogonally diagonalize the matrix by finding an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\vec{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{u_1} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Question 2

Orthogonally diagonalize the matrix by finding an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\vec{v_1} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u_1} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \vec{u_3} = \vec{v_3}$$

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Question 3

Find a spectral decomposition of the matrix.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$\lambda_1 = 2 \quad \lambda_2 = 4$$

$$\vec{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{q_1} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \vec{q_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_1 \vec{q_1}\vec{q_1}^T = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$\lambda_2 \vec{q_2}\vec{q_2}^T = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Question 4

Find a spectral decomposition of the matrix.

$$A = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

$$\lambda_1 = -2 \quad \lambda_2 = 4 \quad \lambda_3 = 6$$

$$\vec{v_1} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{q_1} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \vec{q_2} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \vec{q_3} = \vec{v_3}$$

$$\lambda_1 \vec{q_1} \vec{q_1}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\lambda_2 \vec{q_2} \vec{q_2}^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$\lambda_2 \vec{q_3} \vec{q_3}^T = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Question 5

Find a symmetric 2×2 matrix with eigenvalues λ_1 and λ_2 and corresponding orthogonal eigenvectors $\vec{v_1}$ and $\vec{v_2}$.

$$\lambda_1 = -1 \quad \lambda_2 = 4$$

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{q_1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \vec{q_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_1 \vec{q_1} \vec{q_1}^T = \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

$$\lambda_2 \vec{q_2} \vec{q_2}^T = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} \frac{3}{2} & -\frac{5}{2} \\ -\frac{5}{2} & \frac{3}{2} \end{bmatrix}$$

Question 6

Evaluate the quadratic form $f(\vec{x}) = \vec{x}^T A\vec{x}$ for the given A and $\vec{x}$.

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned} f(\vec{x}) &= 3x^2 + xy + xy + 4y^2 \\ &= 3x^2 + 2xy + 4y^2 \end{aligned}$$

Question 7

Evaluate the quadratic form $f(\vec{x}) = \vec{x}^T A\vec{x}$ for the given A and $\vec{x}$.

$$A = \begin{bmatrix} 3 & -2 \\ -2 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 1 \\ 5 \end{bmatrix} A \qquad = 3x^2 - 4xy + 4y^2$$

$$\begin{aligned} &= 3(1)^2 - 4(1)(5) + 4(5)^2 \\ &= 83 \end{aligned}$$

Question 8

Evaluate the quadratic form $f(\vec{x}) = \vec{x}^T A\vec{x}$ for the given A and $\vec{x}$.

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & 1 \\ -3 & 1 & 3 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{aligned} f(\vec{x}) &= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} x - 3z \\ 2y + z \\ -3x + y + 3z \end{bmatrix} \\ &= x^2 - 3xz + 2y^2 + yz - 3xz + yz + 3z^2 \\ &= x^2 + 2y^2 + 3z^2 + 2yz - 6xz \end{aligned}$$

Question 9

Find the symmetric matrix A associated with the given quadratic form.

$$x_1^2 + 4x_2^2 + 6x_1x_2$$

$$\begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$$

Question 10

Diagonalize the quadratic form by finding an orthogonal matrix Q such that the change of variable $\vec{x} = Q\vec{y}$ transforms the given form into one with no cross-product terms. Given Q and the new quadratic form

$f(\vec{y})$.

$$
\begin{aligned}
f(x_1, x_2) &= 4x_1^2 + 7x_2^2 - 4x_1x_2 \\
A &= \begin{bmatrix} 4 & -2 \\ -2 & 7 \end{bmatrix} \\
\lambda_1 &= 3 \quad \lambda_2 = 8 \\
\vec{v_1} &= \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \\
\vec{q_1} &= \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \quad \vec{q_2} = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \\
Q &= \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} \\
D &= \begin{bmatrix} 3 & 0 \\ 0 & 8 \end{bmatrix} \\
f(y_1, y_2) &= 3y_1^2 + 8y_2^2
\end{aligned}
$$

Question 11

Use a rotation of axes to put the conic in standard position.

$$
\begin{aligned}
x^2 + xy + y^2 &= 9 \\
A &= \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} \\
\lambda_1 &= \frac{1}{2} \quad \lambda_2 = \frac{3}{2} \\
D &= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \\
\frac{1}{2}x' + \frac{3}{2}y' &= 9
\end{aligned}
$$

Question 12

Determine whether the given set, together with specified operations of addition and scalar multiplication, is a vector space. If it is not, select all of the axioms that fail to hold.

The set of all vectors in $\mathbb{R}^2$ of the form $\begin{bmatrix} -3x \\ -3x \end{bmatrix}$, with the usual vector addition and scalar multiplication.

All of the axioms hold, so the given set is a vector space.

Question 13

Determine whether the given set, together with specified operations of addition and scalar multiplication, is a vector space. If it is not, select all of the axioms that fail to hold.

The set of all vectors in $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ with $x \geq 0, y \geq 0$, with the usual vector addition and scalar multiplication.

1. For each $\vec{u}$ in V, there is an element $-\vec{u}$ in V such that $\vec{u} + (-\vec{u}) = \vec{0}$.
2. $c\vec{u}$ is in V.

Question 14

Determine whether the given set, together with specified operations of addition and scalar multiplication, is a vector space. If it is not, select all of the axioms that fail to hold.

$\mathbb{R}^2$, with the usual addition but scalar multiplication defined by

$$c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ cy \end{bmatrix}$$

?

Question 15

Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold.

(a) If $\vec{u}$ and $\vec{v}$ are in W, then $\vec{u} + \vec{v}$ is in W.

(b) If $\vec{u}$ is in W and c is a scalar, then $c\vec{u}$ is in W.

Use the theorem above to determine whether W is a subspace of V.

$$V = \mathbb{R}^3, W = \left\{ \begin{bmatrix} a \\ b \\ |a| \end{bmatrix} \right\}$$

W is not a subspace of V.

Question 16

Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold.

(a) If $\vec{u}$ and $\vec{v}$ are in W, then $\vec{u} + \vec{v}$ is in W.

(b) If $\vec{u}$ is in W and c is a scalar, then $c\vec{u}$ is in W.

Use the theorem above to determine whether W is a subspace of V.

$$V = M_{22}, W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : ad \geq bc \right\}$$

W is not a subspace of V.

Question 17

Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold.

(a) If $\vec{u}$ and $\vec{v}$ are in W, then $\vec{u} + \vec{v}$ is in W.

(b) If $\vec{u}$ is in W and c is a scalar, then $c\vec{u}$ is in W.

Use the theorem above to determine whether W is a subspace of V.

$$V = M_{nn}, W = \{A \text{ in } M_{nn} : \det(A) = 1\}$$

W is not a subspace of V.

Question 18

Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold.

(a) If $\vec{u}$ and $\vec{v}$ are in W, then $\vec{u}+\vec{v}$ is in W.

(b) If $\vec{u}$ is in W and c is a scalar, then $c\vec{u}$ is in W.

Use the theorem above to determine whether W is a subspace of V.

$$V = P_2, W = \{bx + cx^2\}$$

W is a subspace of V.

Question 19

Let V be a vector space and let W be a nonempty subset of V. Then W is a subspace of V if and only if the following conditions hold.

(a) If $\vec{u}$ and $\vec{v}$ are in W, then $\vec{u}+\vec{v}$ is in W.

(b) If $\vec{u}$ is in W and c is a scalar, then $c\vec{u}$ is in W.

Use the theorem above to determine whether W is a subspace of V.

$$V = P_2, W = \{a + bx + cx^2 : abc = 0\}$$

W is not a subspace of V.

Question 20

Let $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$. Determine whether C is in $span(A, B)$.

$$C = \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix}$$

C is not in $span(A, B)$.

Question 21

Let $p(x) = 1 - 2x, q(x) = x - x^2, r(x) = -2 + 3x + x^2$. Determine whether $s(x)$ is in $span(p(x), q(x), r(x))$.

$$\begin{aligned} s(x) &= 4 + x + x^2 \\ ap(x) + bq(x) + cr(x) &= s(x) \\ a - 2ax + bx - bx^2 - 2c + 3cx + cx^2 &= 4 + x + x^2 \\ (a - 2c) + (-2a + b + 3c)x + (-b + c)x^2 &= 4 + x + x^2 \\ \left[\begin{array}{ccc|c} 1 & 0 & -2 & 4 \\ -2 & 1 & 3 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right] &\rightarrow \text{inconsistent} \end{aligned}$$

Since the system is inconsistent, $s(x)$ is not in the span.

Question 22

Test the set of matrices for linear independence in M_{22}. If it is linearly dependent, express one of the matrices as a linear combination of the others.

$$\left\{ A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \\ 5 & 4 \end{bmatrix} \right\}$$

$$c_1 A + c_2 B + c_3 C = 0$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ -1 & 0 & 4 & 0 \end{array}\right] \to \text{trivial}$$

This system has only the trivial solution, so the matrices are linearly independent.

Question 23

Test the set of matrices for linear dependence in M_{22}. If it is linearly dependent, express one of the matrices as a linear combination of the others.

$$\left\{ A = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 \\ 1 & 1 \end{bmatrix}, C = \begin{bmatrix} 0 & 2 \\ -3 & 1 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 \\ -1 & 7 \end{bmatrix} \right\}$$

$$c_1 A + c_2 B + c_3 C + c_3 D = 0$$

$$\left[\begin{array}{cccc|c} -1 & 3 & 0 & -1 & 0 \\ 1 & 0 & 2 & 0 & 0 \\ -2 & 1 & -3 & -1 & 0 \\ 2 & 1 & 1 & 7 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$c_1 = -4c_4 \quad c_2 = -c_4 \quad c_3 = 2c_4$$

Question 24

Test the set of polynomials for linear independence. If it is linearly dependent, express one of the polynomials as a linear combination of the others.

$$\{f(x) = 3 + x, g(x) = 3 + x^2, h(x) = 3 - x + x^2\}$$

$$\begin{aligned} af(x) + bg(x) + ch(x) &= 0 \\ 3a + ax + 3b + bx^2 + 3c - cx + cx^2 &= 0 \\ (3a + 3b + 3c) + (a - c)x + (b + c)x^2 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 3 & 3 & 3 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \to \text{trivial}$$

This system has only the trivial solution, so the matrices are linearly independent.

Question 25

Test the set of polynomials for linear independence. If it is linearly dependent, express one of the polynomials as a linear combination of the others.

$$\{f(x) = x, g(x) = 2x - x^2, h(x) = 2x + 3x^2\}$$

$$\begin{aligned} af(x) + bg(x) + ch(x) &= 0 \\ ax + 2bx - bx^2 + 2cx + 3cx^2 &= 0 \\ (a + 2b + 2c)x + (-b + 3c)x^2 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & -1 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 8 & 0 \\ 0 & 1 & -3 & 0 \end{array}\right]$$

$$a = -8c \quad b = 3c$$

Question 26

Determine whether the set $\mathbb{B}$ is a basis for the vector space V.

$$V = M_{22}, \mathbb{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \right\}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \rightarrow \text{trivial}$$

The set $\mathbb{B}$ is linearly independent but does not span the vector space, so it is not a basis.

Question 27

Determine whether the set $\mathbb{B}$ is a basis for the vector space V.

$$V = M_{22}, \mathbb{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \text{trivial}$$

The set $\mathbb{B}$ is linearly independent and spans the vector space (a solution exists), so it is a basis.

Question 28

Determine whether the set $\mathbb{B}$ is a basis for the vector space V.

$$V = P_2, \mathbb{B} = \{x, 4 + x, x - x^2\}$$

$$\begin{aligned} (ax) + (4b + bx) + (cx - cx^2) &= 0 \\ (4b) + (a + b + c)x + (-c)x^2 &= 0 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 0 & 4 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right]$$

The set $\mathbb{B}$ is linearly independent and spans the vector space (a solution exists), so it is a basis.

Question 29

Find the coordinate vector of $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ with respect to the basis $\mathbb{B} = \{E_{22}, E_{21}, E_{12}, E_{11}\}$ of M_{22}.

$$A = 6E_{22} + 5E_{21} + 4E_{12} + 3E_{11}$$

Question 30

Find the coordinate vector of $p(x) = 6 - x + 5x^2$ with respect to the basis $\mathbb{B} = \{1, 1+x, -1+x^2\}$ of P_2.

$$(a) + (b + bx) + (-c + cx^2) = 6 - x + 5x^2$$
$$(a + b - c) + (b)x + (c)x^2 = 6 - x + 5x^2$$
$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 6 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{array}\right]$$
$$[p]_{\mathbb{B}} = \begin{bmatrix} 12 \\ -1 \\ 5 \end{bmatrix}$$

Question 31

Find a formula for the dimension of the vector space V of symmetric $n \times n$ matrices.

$$dim(V) = \frac{n(n+1)}{2}$$

The basis of an $n \times n$ matrix

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech