Advanced Linear Algebra: Homework 4

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Question 1

Determine if the given vectors form an orthogonal set.

$$\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\0\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\0\\1\\2\end{bmatrix} = 0 \quad \begin{bmatrix}1\\0\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\\0\end{bmatrix} = 0 \quad \begin{bmatrix}1\\0\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}0\\-1\\1\\1\end{bmatrix} = 0$$

$$\begin{bmatrix}-1\\0\\1\\2\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\\0\end{bmatrix} = 0 \quad \begin{bmatrix}-1\\0\\1\\2\end{bmatrix} \cdot \begin{bmatrix}0\\-1\\1\\1\end{bmatrix} = 3$$

Question 2

Do the given vectors form an orthogonal basis for $\mathbb{R}^3$?

$$\vec{v_1} = \begin{bmatrix}1\\0\\-1\end{bmatrix}, \quad \vec{v_2} = \begin{bmatrix}2\\4\\2\end{bmatrix}, \quad \vec{v_3} = \begin{bmatrix}3\\-3\\3\end{bmatrix}$$

$$\vec{v_1} \cdot \vec{v_2} = 0 \quad \vec{v_1} \cdot \vec{v_3} = 0$$
$$\vec{v_2} \cdot \vec{v_3} = 0$$

Let $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and let $\vec{w}$ be any vector in W. Then the unique scalars $c_1, \dots, c_k$, such that

$$\vec{w} = c_1\vec{v_1} + \cdots + c_k\vec{v_k}$$
$$c_i = \frac{\vec{w} \cdot \vec{v_i}}{\vec{v_i} \cdot \vec{v_i}} \quad i = 1, \dots, k$$

Use the theorem to express $\vec{w}$ as a linear combination of the above basis vectors. Give the coordinate center $[\vec{w}]_{\mathbb{B}}$ of $\vec{w}$ with respect to the basis $\vec{b} = \{\vec{v_1}, \vec{v_2}\}$ of $\mathbb{R}^3$.

$$\begin{aligned}
\vec{w} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
c_1 &= \frac{\vec{w} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} = \frac{1+0-1}{1+0+1} = 0 \\
c_2 &= \frac{\vec{w} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} = \frac{2+4+2}{4+16+4} = \frac{1}{3} \\
c_3 &= \frac{\vec{w} \cdot \vec{v_3}}{\vec{v_3} \cdot \vec{v_3}} = \frac{3-3+3}{9+9+9} = \frac{1}{9} \\
[\vec{w}]_{\mathbb{B}} &= \begin{bmatrix} 0 \\ \frac{1}{3} \\ \frac{1}{9} \end{bmatrix}
\end{aligned}$$

Question 3

Determine whether the given matrix is orthogonal.

$$\begin{aligned}
Q &= \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{4} & \frac{1}{3} \\ -\frac{1}{2} & 0 & \frac{2}{3} \end{bmatrix} \\
Q^T Q &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{4} & \frac{1}{3} \\ -\frac{1}{2} & 0 & \frac{2}{3} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{4} & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & \frac{2}{3} \end{bmatrix}
\end{aligned}$$

The matrix is not orthogonal.

Question 4

Determine whether the given matrix is orthogonal.

$$\begin{aligned}
Q &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
Q^T Q &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
&= I_4 \\
Q^{-1} &= Q^T
\end{aligned}$$

This matrix is orthogonal.

Question 5

The following facts can be proven.

(i) An orthogonal 2×2 matrix must have the form

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \text{ or } \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$$

where $\begin{bmatrix} a \\ b \end{bmatrix}$ is a unit vector.

(ii) Every orthogonal 2×2 matrix is of the form

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \text{ or } \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

(iii) Every orthogonal 2×2 matrix corresponds to either a rotation or a reflection $\mathbb{R}^2$.

(iv) An orthogonal 2×2 matrix corresponds to a rotation in $\mathbb{R}^2$ if $\det(Q) = 1$ and a reflection in $\mathbb{R}^2$ if $\det(Q) = -1$.

Use the above facts to determine whether the given orthogonal matrix represents a rotation or a reflection.

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\det(Q) = 1 \quad \theta = \frac{\pi}{4}$$

Question 6

Find the orthogonal complement $W^\perp$ of W and give a basis for $W^\perp$.

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x + y - z = 0 \right\}$$

$$x = z - y \quad y = s \quad z = t$$

$$W = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$W^\perp = span \left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right)$$

Question 7

Find the orthogonal complement $W^{\perp}$ of W and give a basis for $W^{\perp}$

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = \frac{1}{2}t, y = -\frac{1}{2}t, z = 3t \right\}$$
$$= t \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{0}$$
$$x_1 - x_2 + 6x_3 = 0$$
$$x_1 = x_2 - 6x_3 \quad x_2 = s \quad x_3 = t$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix}$$
$$W^{\perp} = span \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix} \right)$$

Question 8

Find a basis for the row space of A.

$$A = \begin{bmatrix} 1 & -1 & 3 \\ -3 & 1 & -5 \\ 0 & 1 & -2 \\ 5 & -3 & 11 \end{bmatrix}$$
$$rref(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$row(A) = span \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \right)$$
$$A\vec{x} = 0$$
$$\vec{x_1} = -x_3 \quad \vec{x_2} = 2\vec{x_3} \quad \vec{x_3} = s$$
$$\vec{x} = s \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$
$$null(A) = span \left(\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right)$$

Question 9

Find a basis for the column space of A.

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 2 & 1 \\ 0 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$rref(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$col(A) = span\left(\begin{bmatrix} 1 \\ 5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ -1 \end{bmatrix}\right)$$

Find a basis for the null space of A^T for the given matrix. Verify that every vector in $col(A)$ is orthogonal to every vector in $null(A^T)$.

$$A^T\vec{x} = \vec{0}$$

$$\left[\begin{array}{cccc|c} 1 & 5 & 0 & -1 & 0 \\ -1 & 2 & 1 & -1 & 0 \\ 3 & 1 & -2 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -\frac{5}{7} & \frac{3}{7} & 0 \\ 0 & 1 & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = \frac{5}{7}x_3 - \frac{3}{7}x_4 \quad x_2 = -\frac{1}{7}x_3 + \frac{2}{7}x_4 \quad x_3 = s \quad x_4 = t$$

$$\vec{x} = s\begin{bmatrix} \frac{5}{7} \\ -\frac{1}{7} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -\frac{3}{7} \\ \frac{2}{7} \\ 0 \\ 1 \end{bmatrix}$$

$$null(A^T) = span\left(\begin{bmatrix} \frac{5}{7} \\ -\frac{1}{7} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{7} \\ \frac{2}{7} \\ 0 \\ 1 \end{bmatrix}\right)$$

Question 10

Let W be the subspace spanned by the given vectors. Find a basis for $W^{\perp}$.

$$\vec{w_1} = \begin{bmatrix} 1 \\ -1 \\ 4 \\ -2 \end{bmatrix} \quad \vec{w_2} = \begin{bmatrix} 0 \\ 1 \\ -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 4 & -2 \\ 0 & 1 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \vec{0}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 4 & -2 & 0 \\ 0 & 1 & -3 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & -3 & 1 & 0 \end{array}\right]$$

$$x_1 = -x_3 + x_4 \quad x_2 = 3x_3 - x_4 \quad x_3 = s \quad x_4 = t$$

$$\vec{x} = s \begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$W^{\perp} = span \left(\begin{bmatrix} -1 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Question 11

Let W be the subspace spanned by the given vectors. Find a basis for $W^{\perp}$.

$$\vec{w_1} = \begin{bmatrix} 4 \\ -5 \\ 12 \\ 7 \end{bmatrix} \quad \vec{w_2} = \begin{bmatrix} 2 \\ 2 \\ 6 \\ 2 \end{bmatrix} \quad \vec{w_3} = \begin{bmatrix} -2 \\ -8 \\ -6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -5 & 12 & 7 & 0 \\ 2 & 2 & 6 & 2 & 0 \\ -2 & -8 & -6 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$$

$$\left[\begin{array}{cccc|c} 4 & -5 & 12 & 7 & 0 \\ 2 & 2 & 6 & 2 & 0 \\ -2 & -8 & -6 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 3 & \frac{4}{3} & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_1 = -3x_3 - \frac{4}{3}x_4 \quad x_2 = \frac{1}{3}x_4 \quad x_3 = s \quad x_4 = t$$

$$\vec{x} = s \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{4}{3} \\ \frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

$$W^{\perp} = span \left(\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{4}{3} \\ \frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right)$$

Question 12

Find the orthogonal projection of $\vec{v}$ onto the subspace W spanned by the vectors $\vec{u_i}$. (You may assume that the vectors $\vec{u_i}$ are orthogonal.)

$$\vec{v} = \begin{bmatrix} 7 \\ -4 \end{bmatrix} \quad \vec{u_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned} \text{proj}_W \vec{v} &= \left(\frac{\vec{u_1} \cdot \vec{v}}{\vec{u_1} \cdot \vec{u_1}} \right) \vec{u_1} \\ &= \frac{7-4}{1+1} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \end{bmatrix} \end{aligned}$$

Question 13

Find the orthogonal projection of $\vec{v}$ onto the subspace W spanned by the vectors $\vec{u_i}$.

$$\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{u_1} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{aligned} \text{proj}_W \vec{v} &= \sum_{i=1}^{2} \left(\frac{\vec{u_i} \cdot \vec{v}}{\vec{u_i} \cdot \vec{u_i}} \right) \vec{u_i} \\ &= \frac{2-4+3}{4+4+1} \vec{u_1} + \frac{-1+2+12}{1+1+16} \vec{u_2} \\ &= \frac{1}{9} \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} + \frac{13}{18} \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{9}{18} \\ \frac{9}{18} \\ \frac{54}{18} \end{bmatrix} \end{aligned}$$

Question 14

Find the orthogonal projection of $\vec{v}$ onto the subspace W spanned by the vectors $\vec{u_i}$.

$$\vec{v} = \begin{bmatrix} 6 \\ -4 \\ 8 \\ -6 \end{bmatrix} \quad \vec{u_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \vec{u_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\text{proj}_W \vec{v} &= \sum_{i=1}^{2} \left(\frac{\vec{u_i} \cdot \vec{v}}{\vec{u_i} \cdot \vec{u_i}} \right) \vec{u_i} \\
&= \frac{6-4}{1+1} \vec{u_1} + \frac{6+4-8-6}{1+1+1+1} \vec{u_2} + \frac{8-6}{1+1} \vec{u_3} \\
&= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 2 \\ 2 \\ 0 \end{bmatrix}
\end{aligned}$$

Question 15

Find the orthogonal decomposition of $\vec{v}$ with respect to W.

$$\vec{v} = \begin{bmatrix} 2 \\ -2 \end{bmatrix} \quad W = span\left(\begin{bmatrix} 1 \\ 4 \end{bmatrix} \right)$$

$$\begin{aligned}
\text{proj}_W \vec{v} &= \sum_{i=1}^{1} \left(\frac{\vec{u_i} \cdot \vec{v}}{\vec{u_i} \cdot \vec{u_i}} \right) \vec{u_i} \\
&= \frac{2-8}{1+16} \begin{bmatrix} 1 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} -\frac{6}{17} \\ -\frac{24}{17} \end{bmatrix} \\
\text{perp}_W \vec{v} &= \vec{v} - \text{proj}_W \vec{v} \\
&= \begin{bmatrix} \frac{40}{17} \\ -\frac{10}{17} \end{bmatrix}
\end{aligned}$$

Question 16

Find the orthogonal decomposition of $\vec{v}$ with respect to W.

$$\vec{v} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \quad W = span\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right)$$

$$\begin{aligned}
\text{proj}_W \vec{v} &= \sum_{i=1}^{2} \left(\frac{\vec{u_i} \cdot \vec{v}}{\vec{u_i} \cdot \vec{u_i}} \right) \vec{u_i} \\
&= \frac{4-4+3}{1+4+1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + \frac{4+2+3}{1+1+1} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} \frac{7}{2} \\ -2 \\ \frac{7}{2} \end{bmatrix} \\
\text{perp}_W \vec{v} &= \vec{v} - \text{proj}_W \vec{v} \\
&= \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}
\end{aligned}$$

Question 17

The given vectors form a basis for a subspace W of $\mathbb{R}^3$. Apply the Gram-Schmidt Process to obtain an orthogonal basis. Then normalize this basis to obtain an orthonormal basis.

$$\vec{x_1} = \begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix} \quad \vec{x_2} = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix} \quad \vec{x_3} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

$$\begin{aligned}
\vec{v_1} &= \vec{x_1} \quad W_1 = span(\vec{v_1}) \\
\vec{v_2} &= \vec{x_2} - \text{proj}_{W_1} \vec{x_2} \\
&= \vec{x_2} - \sum_{i=1}^{1} \left(\frac{\vec{u_i} \cdot \vec{x_2}}{\vec{u_i} \cdot \vec{u_i}} \right) \vec{u_i} \\
&= \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad W_2 = span(\vec{v_1}, \vec{v_2}) \\
\vec{v_3} &= \vec{x_3} - \text{proj}_{W_2} \vec{x_3} \\
&= \vec{x_3} - \sum_{i=1}^{2} \left(\frac{\vec{u_i} \cdot \vec{x_3}}{\vec{u_i} \cdot \vec{u_i}} \right) \vec{u_i} \\
&= \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \\
\mathbb{B} &= span\left(\begin{bmatrix} 2 \\ -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right)
\end{aligned}$$

Question 18

The given vectors form a basis for a subspace W of $\mathbb{R}^3$. Apply the Gram-Schmidt Process to obtain an orthogonal basis. Then normalize this basis to obtain an orthonormal basis.

$$\vec{x_1} = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix} \quad \vec{x_2} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$$

$$\vec{v_1} = \vec{x_1} \quad W_1 = span(\vec{v_1})$$

$$\begin{aligned} \vec{v_2} &= \vec{x_2} - \text{proj}_{W_1}\vec{x_2} \\ &= \vec{x_2} - \sum_{i=1}^{1} \left(\frac{\vec{u_i} \cdot \vec{x_2}}{\vec{u_i} \cdot \vec{u_i}} \right) \vec{u_i} \\ &= \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \mathbb{B} &= span\left(\begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix} \right) \\ &= span\left(\begin{bmatrix} -\frac{2}{\sqrt{8}} \\ -\frac{2}{\sqrt{8}} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{\sqrt{2}}{6} \\ \frac{\sqrt{2}}{6} \\ \frac{2\sqrt{2}}{3} \end{bmatrix} \right) \end{aligned}$$

Question 19

The given vectors form a basis for a subspace W of $\mathbb{R}^3$. Apply the Gram-Schmidt Process to obtain an orthogonal basis. Then normalize this basis to obtain an orthonormal basis.

$$\vec{x_1} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix} \quad \vec{x_2} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 4 \end{bmatrix} \quad \vec{x_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v_1} = \vec{x_1} \quad W_1 = span(\vec{v_1})$$

$$\begin{aligned} \vec{v_2} &= \vec{x_2} - \text{proj}_{W_1}\vec{x_2} \\ &= \vec{x_2} - \sum_{i=1}^{1} \left(\frac{\vec{u_i} \cdot \vec{x_2}}{\vec{u_i} \cdot \vec{u_i}} \right) \vec{u_i} \\ &= \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix} \quad W_2 = span(\vec{v_2}, \vec{v_2}) \end{aligned}$$

$$\begin{aligned} \vec{v_3} &= \vec{x_3} - \text{proj}_{W_2}\vec{x_3} \\ &= \vec{x_3} - \sum_{i=1}^{2} \left(\frac{\vec{u_i} \cdot \vec{x_3}}{\vec{u_i} \cdot \vec{u_i}} \right) \vec{u_i} \\ &= \begin{bmatrix} \frac{1}{5} \\ \frac{7}{5} \\ \frac{3}{5} \\ \frac{1}{5} \end{bmatrix} \end{aligned}$$

$$\mathbb{B} = span\left(\begin{bmatrix} 2 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{3}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{5} \\ \frac{7}{5} \\ \frac{3}{5} \\ \frac{1}{5} \end{bmatrix} \right)$$

Question 20

Use the Gram-Schmidt Process to find an orthogonal basis for the column space of the matrix.

$$A = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\vec{v_1} = \vec{x_1} \quad W_1 = span(\vec{v_1})$$

$$\vec{v_2} = \vec{x_2} - \text{proj}_{W_1}\vec{x_2}$$

$$= \vec{x_2} - \sum_{i=1}^{1} \left(\frac{\vec{u_i} \cdot \vec{x_2}}{\vec{u_i} \cdot \vec{u_i}} \right) \vec{u_i}$$

$$= \begin{bmatrix} -1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \qquad W_2 = span(\vec{v_2}, \vec{v_2})$$

$$\vec{v_3} = \vec{x_3} - \text{proj}_{W_2}\vec{x_3}$$

$$= \vec{x_3} - \sum_{i=1}^{2} \left(\frac{\vec{u_i} \cdot \vec{x_3}}{\vec{u_i} \cdot \vec{u_i}} \right) \vec{u_i}$$

$$= \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix}$$

$$\mathbb{B} = span\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{2}{3} \end{bmatrix} \right)$$

Question 21

Find an orthogonal basis for $\mathbb{R}^4$ that contains the following vectors.

$$\begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix} \quad \vec{x_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \quad \vec{x_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{v_4} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

If we apply the Gram-Schmidt process, we get a basis

$$\mathbb{B} = span\left(\begin{bmatrix} 1 \\ 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -\frac{3}{11} \\ \frac{2}{11} \\ \frac{3}{11} \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ 0 \\ -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \right)$$

Question 22

Find a QR factorization of the matrix.

$$A = \begin{bmatrix} 0 & 5 & 3 \\ 1 & 0 & 3 \\ 1 & 5 & 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 5 & 2 \\ 1 & -\frac{5}{2} & 2 \\ 1 & \frac{5}{2} & -2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{3}}{3} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} \sqrt{2} & \frac{5}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \frac{5\sqrt{6}}{2} & \frac{\sqrt{6}}{2} \\ 0 & 0 & 2\sqrt{3} \end{bmatrix}$$

Question 23

The columns of Q were obtained by applying the Gram-Schmidt Process to columns of A. Find the upper triangular matrix R such that $A = QR$.

$$A = \begin{bmatrix} 5 & 5 & 5 \\ 10 & 4 & -1 \\ -10 & -7 & 6 \end{bmatrix}$$

$$Q = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$R = Q^T A = \begin{bmatrix} 15 & 9 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

Question 24

Orthogonally diagonalize the matrix by finding an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$\lambda_1 = 1 \quad \lambda_2 = 5$$

$$\vec{v_1} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\vec{u_1} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$Q = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Question 25

Orthogonally diagonalize the matrix by finding an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$\lambda_1 = -2 \quad \lambda_2 = 4 \quad \lambda_3 = 5$$

$$D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\vec{v_1} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{u_1} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \quad \vec{u_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech