Advanced Linear Algebra: Homework 3

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Question 1

A matrix A is given along with an iterate x_5.

$$A = \begin{bmatrix} 7 & 2 \\ -6 & -1 \end{bmatrix}, x_5 = \begin{bmatrix} 6249 \\ -6247 \end{bmatrix}$$

(a) Use these data to approximate a dominant eigenvector whose first component is 1 and a corresponding dominant eigenvalue.

$$\begin{aligned}
\vec{x_6} &= A\vec{x_5} \\
&= \begin{bmatrix} 7 & 2 \\ -6 & -1 \end{bmatrix} \begin{bmatrix} 6249 \\ -6247 \end{bmatrix} \\
&= \begin{bmatrix} 31249 \\ -31247 \end{bmatrix} \\
\vec{x_6} &\approx \lambda_1 \vec{x_5} \\
\begin{bmatrix} 31249 \\ -31247 \end{bmatrix} &\approx \lambda_1 \begin{bmatrix} 6249 \\ -6247 \end{bmatrix} \\
\lambda_1 &\approx 5.001
\end{aligned}$$

(b) Compare your approximate eigenvalue in part (a) with the actual dominant eigenvalue, λ.

$$\begin{aligned}
\det(A - \lambda I) &= \begin{vmatrix} 7-\lambda & 2 \\ -6 & -1-\lambda \end{vmatrix} \\
&= (7-\lambda)(-1-\lambda) - (-12) \\
&= -7 - 7\lambda + \lambda + \lambda^2 + 12 \\
&= \lambda^2 - 6\lambda + 5 \\
&= (\lambda - 5)(\lambda - 1) \\
&= 0 \\
\lambda = 5 &\quad \lambda = 1
\end{aligned}$$

Question 2

Use the power method to approximate the dominant eigenvalue and eigenvector of A. Use the given initial vector v_0, the specified number of iterations k, and three-decimal-place accuracy.

$$A = \begin{bmatrix} 17 & 12 \\ 6 & 3 \end{bmatrix}, x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, k = 5$$

$$\vec{x_1} = \begin{bmatrix} 17 & 12 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 29 \\ 9 \end{bmatrix}$$

$$\vec{x_2} = \begin{bmatrix} 17 & 12 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 29 \\ 9 \end{bmatrix} = \begin{bmatrix} 601 \\ 201 \end{bmatrix}$$

$$\vec{x_3} = \begin{bmatrix} 17 & 12 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 601 \\ 201 \end{bmatrix} = \begin{bmatrix} 12629 \\ 4209 \end{bmatrix}$$

$$\vec{x_4} = \begin{bmatrix} 17 & 12 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 12629 \\ 4209 \end{bmatrix} = \begin{bmatrix} 265201 \\ 88401 \end{bmatrix}$$

$$\begin{bmatrix} 265201 \\ 88401 \end{bmatrix} = \lambda_1 \begin{bmatrix} 12629 \\ 4209 \end{bmatrix}$$

$$\lambda_1 \approx 21$$

Question 3

Draw the Gerschgorin disks for the given matrix.

$$\begin{bmatrix} 1 & 1 & 0 \\ \frac{1}{2} & 5 & \frac{1}{2} \\ 1 & 0 & 6 \end{bmatrix}$$

The solution is three circles on the imaginary-real plane centered at $(1, 0), (5, 0), (6, 0)$ with radius 1 by rows. By columns, the solution is three circles on the imaginary-real plane centered at $(1, 0), (5, 0), (6, 0)$ with radii $1.5, 1, 0.5$ respectively.

Question 4

Find the general solution to the given system of differential equations.

$$
\begin{aligned}
x' &= x + 4y \quad x(0) = 0 \\
y' &= 3x + 2y \quad y(0) = 7 \\
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \\
&= (PDP^{-1})\begin{bmatrix} x \\ y \end{bmatrix} \\
D &= \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} \\
P &= \begin{bmatrix} -\frac{4}{3} & 1 \\ 1 & 1 \end{bmatrix} \\
\frac{\mathrm{d}P^{-1}\vec{x}}{\mathrm{d}t} &= D(P^{-1}\vec{x}) \quad \vec{z} = P^{-1}\vec{x} \\
\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} &= \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \\
\frac{\mathrm{d}z_1}{\mathrm{d}t} &= -2z_1 \quad z_1 = c_1 \mathrm{e}^{-2t} \\
\frac{\mathrm{d}z_2}{\mathrm{d}t} &= 5z_2 \quad z_2 = c_2 \mathrm{e}^{5t} \\
\vec{x} &= P\vec{z} \\
&= \begin{bmatrix} -\frac{4}{3} & 1 \\ 1 & 1 \end{bmatrix}\vec{z} \\
&= \begin{bmatrix} -\frac{4}{3}c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{5t} \\ c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{5t} \end{bmatrix} \\
x(0) &= 0 = -\frac{4}{3}c_1 + c_2 \\
y(0) &= 7 = c_1 + c_2 \\
c_1 &= 3 \quad c_2 = 4
\end{aligned}
$$

Question 5

Find the general solution to the given system of differential equations.

$$x' = y - z \quad x(0) = 1$$
$$y' = x + z \quad y(0) = 0$$
$$z' = x + y \quad z(0) = -1$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$
$$= (PDP^{-1})\vec{x}$$
$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$P = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$
$$\frac{\mathrm{d}P^{-1}\vec{x}}{\mathrm{d}t} = D(P^{-1}\vec{x}) \quad P^{-1}\vec{x} = \vec{a}$$
$$\frac{\mathrm{d}\vec{a}}{\mathrm{d}t} = D\vec{a}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$
$$\frac{\mathrm{d}a_1}{\mathrm{d}t} = 0 \quad a_1 = c_1\mathrm{e}^{0t} = c_1$$
$$\frac{\mathrm{d}a_2}{\mathrm{d}t} = -a_2 \quad a_2 = c_2\mathrm{e}^{-t}$$
$$\frac{\mathrm{d}a_3}{\mathrm{d}t} = a_3 \quad a_3 = c_3\mathrm{e}^{t}$$
$$\vec{x} = P\vec{a}$$
$$= \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}\vec{a}$$
$$= \begin{bmatrix} -c_1 - c_2\mathrm{e}^{-t} \\ c_1 + c_2\mathrm{e}^{-t} + c_3\mathrm{e}^{t} \\ c_1 + c_3\mathrm{e}^{t} \end{bmatrix}$$
$$x(0) = 1 = -c_1 - c_2$$
$$y(0) = 0 = c_1 + c_2 + c_3$$
$$z(0) = -1 = c_1 + c_3$$
$$\left[\begin{array}{ccc|c} -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 - \mathrm{e}^{-t} \\ -2 + \mathrm{e}^{-t} + \mathrm{e}^{t} \\ -2 + \mathrm{e}^{t} \end{bmatrix}$$

Question 6

Determine if the vectors form an orthogonal set.

$$\begin{bmatrix}-9\\3\\6\end{bmatrix}, \begin{bmatrix}6\\12\\3\end{bmatrix}, \begin{bmatrix}3\\-3\\6\end{bmatrix}$$

$$\begin{bmatrix}-9\\3\\6\end{bmatrix} \cdot \begin{bmatrix}6\\12\\3\end{bmatrix} = 0$$
$$\begin{bmatrix}-9\\3\\6\end{bmatrix} \cdot \begin{bmatrix}3\\-3\\6\end{bmatrix} = 0$$
$$\begin{bmatrix}6\\12\\3\end{bmatrix} \cdot \begin{bmatrix}3\\-3\\6\end{bmatrix} = 0$$

Question 7

Determine if the given vectors form an orthogonal set.

$$\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\\2\end{bmatrix}, \begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\\1\end{bmatrix}$$

$$\begin{bmatrix}1\\0\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}-1\\0\\1\\2\end{bmatrix} = 0 \quad \begin{bmatrix}1\\0\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\\0\end{bmatrix} = 0 \quad \begin{bmatrix}1\\0\\-1\\1\end{bmatrix} \cdot \begin{bmatrix}0\\-1\\1\\1\end{bmatrix} = 0$$
$$\begin{bmatrix}-1\\0\\1\\2\end{bmatrix} \cdot \begin{bmatrix}1\\1\\1\\0\end{bmatrix} = 0 \quad \begin{bmatrix}-1\\0\\1\\2\end{bmatrix} \cdot \begin{bmatrix}0\\-1\\1\\1\end{bmatrix} = 2$$

Question 8

Do the given vectors form an orthogonal basis for $\mathbb{R}^2$.

$$\vec{v_1} = \begin{bmatrix}6\\-2\end{bmatrix} \vec{v_2} = \begin{bmatrix}1\\3\end{bmatrix}$$
$$\vec{v_1} \cdot \vec{v_2} = 0$$

You are given the theorem below.
Let $\{\vec{v_1}, \vec{v_2}, \dots, \vec{v_k}\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and let $\vec{w}$ be any vector in W. Then the unique scalars $c_1, \dots, c_k$, such that

$$\vec{w} = c_1\vec{v_1} + \cdots + c_k\vec{v_k}$$
$$c_i = \frac{\vec{w} \cdot \vec{v_i}}{\vec{v_i} \cdot \vec{v_i}} \quad i = 1, \dots, k$$

Use the theorem to express $\vec{w}$ as a linear combination of the above basis vectors. Give the coordinate center $[\vec{w}]_{\mathbb{B}}$ of $\vec{w}$ with respect to the basis $\vec{b} = \{\vec{v_1}, \vec{v_2}\}$ of $\mathbb{R}^2$.

$$\begin{aligned}
\vec{w} &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\
c_1 &= \frac{\vec{w} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} \\
&= \frac{10}{40} = \frac{1}{4} \\
c_2 &= \frac{\vec{w} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} \\
&= \frac{-5}{10} = -\frac{1}{2} \\
[\vec{w}]_{\mathbb{B}} &= \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{4} \end{bmatrix}
\end{aligned}$$

Question 9

Do the given vectors form an orthogonal basis for $\mathbb{R}^3$?

$$\vec{v_1} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \vec{v_2} = \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} \vec{v_3} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$$

$$\begin{aligned}
\vec{v_1} \cdot \vec{v_2} &= 0 \\
\vec{v_1} \cdot \vec{v_3} &= 0 \\
\vec{v_2} \cdot \vec{v_3} &= 0
\end{aligned}$$

Let $\{\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and let $\vec{w}$ be any vector in W. Then the unique scalars $c_1, \ldots, c_k$, such that

$$\begin{aligned}
\vec{w} &= c_1\vec{v_1} + \cdots + c_k\vec{v_k} \\
c_i &= \frac{\vec{w} \cdot \vec{v_i}}{\vec{v_i} \cdot \vec{v_i}} \quad i = 1, \ldots, k
\end{aligned}$$

Use the theorem to express $\vec{w}$ as a linear combination of the above basis vectors. Give the coordinate center $[\vec{w}]_{\mathbb{B}}$ of $\vec{w}$ with respect to the basis $\vec{b} = \{\vec{v_1}, \vec{v_2}\}$ of $\mathbb{R}^3$.

$$\begin{aligned}
\vec{w} &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
c_1 &= \frac{\vec{w} \cdot \vec{v_1}}{\vec{v_1} \cdot \vec{v_1}} = \frac{0}{2} = 0 \\
c_2 &= \frac{\vec{w} \cdot \vec{v_2}}{\vec{v_2} \cdot \vec{v_2}} = \frac{12}{54} = \frac{2}{9} \\
c_3 &= \frac{\vec{w} \cdot \vec{v_3}}{\vec{v_3} \cdot \vec{v_3}} = \frac{3}{27} = \frac{1}{9} \\
[\vec{w}]_{\mathbb{B}} &= \begin{bmatrix} 0 \\ \frac{2}{9} \\ \frac{1}{9} \end{bmatrix}
\end{aligned}$$

Question 10

Determine whether the given orthogonal set of vectors is orthonormal.

$$\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}$$

$$\left\| \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \right\| = 1$$

$$\left\| \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} \right\| = 1$$

$$\begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \cdot \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} = 0$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech