

Advanced Linear Algebra: Homework 2

Alvin Lin

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Question 1

Show that \vec{v} is an eigenvector of A and find the corresponding eigenvalue, λ .

$$\begin{aligned}A &= \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} & \vec{v} &= \begin{bmatrix} 6 \\ -6 \end{bmatrix} \\A\vec{v} &= \lambda\vec{v} \\ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ -6 \end{bmatrix} &= \lambda \begin{bmatrix} 6 \\ -6 \end{bmatrix} \\ \begin{bmatrix} -6 \\ 6 \end{bmatrix} &= \lambda \begin{bmatrix} 6 \\ -6 \end{bmatrix} \\ \lambda &= -1\end{aligned}$$

Question 2

Show that \vec{v} is an eigenvector of A and find the corresponding eigenvalue, λ .

$$\begin{aligned}A &= \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & 4 & 0 \end{bmatrix} & \vec{v} &= \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \\A\vec{v} &= \lambda\vec{v} \\ \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 3 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} &= \lambda \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} &= \lambda \begin{bmatrix} -4 & 1 & 1 \end{bmatrix} \\ \lambda &= 0\end{aligned}$$

Question 3

Show that λ is an eigenvalue of A and find one eigenvector \vec{v} corresponding to this eigenvalue.

$$\begin{aligned}A &= \begin{bmatrix} 4 & 3 \\ 2 & -1 \end{bmatrix} & \lambda &= 5 \\A\vec{v} &= \lambda\vec{v} \\(A - \lambda I)\vec{v} &= 0 \\ \begin{bmatrix} 4-5 & 3 \\ 2 & -1-5 \end{bmatrix} \vec{v} &= 0 \\ \begin{bmatrix} -1 & 3 & 0 \\ 2 & -6 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ x_1 &= 3x_3 = 3s \\ \vec{x} &= s \begin{bmatrix} 3 \\ 1 \end{bmatrix}\end{aligned}$$

Question 4

Show that λ is an eigenvalue of A and find one eigenvector \vec{v} corresponding to this eigenvalue.

$$\begin{aligned}A &= \begin{bmatrix} 6 & 1 & -1 \\ 1 & 4 & 1 \\ 4 & 2 & 3 \end{bmatrix} & \lambda &= 5 \\A\vec{v} &= \lambda\vec{v} \\(A - \lambda I)\vec{v} &= 0 \\ \begin{bmatrix} 6-5 & 1 & -1 \\ 1 & 4-5 & 1 \\ 4 & 2 & 3-5 \end{bmatrix} \vec{v} &= 0 \\ \begin{bmatrix} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 4 & 2 & -2 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ x_1 &= 0 & x_2 - x_3 &= 0 \\ x_2 &= x_3 = s \\ \vec{v} &= s \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\end{aligned}$$

Question 5

Use the method of Example 4.5 to find all of the eigenvalues λ of the matrix A .

$$A = \begin{bmatrix} 2 & 7 \\ 9 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2 - \lambda & 7 \\ 9 & -\lambda \end{vmatrix} = 0$$

$$(2 - \lambda)(-\lambda) - 63 = 0$$

$$-2\lambda + \lambda^2 - 63 = 0$$

$$\lambda^2 - 2\lambda - 63 = 0$$

$$(\lambda - 9)(\lambda + 7) = 0$$

$$\lambda_1 = 9 \quad \lambda_2 = -7$$

Give bases for each of the corresponding eigenspaces.

$$(A + 7I)\vec{v} = 0$$

$$\begin{bmatrix} 9 & 7 & 0 \\ 9 & 7 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 9 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$9x_1 + 7x_2 = 0$$

$$x_2 = -\frac{9}{7}x_1$$

$$\vec{v} = s \begin{bmatrix} 1 \\ -\frac{9}{7} \end{bmatrix}$$

$$(A - 9I)\vec{v} = 0$$

$$\begin{bmatrix} -7 & 7 & 0 \\ 9 & -9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_2 = s$$

$$\vec{v} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Question 6

Find all of the eigenvalues of the matrix A over the complex numbers \mathbb{C} . Give bases for each of the corresponding eigenspaces.

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 2 - \lambda & -1 \\ 1 & 2 - \lambda \end{vmatrix} = 0$$

$$(2 - \lambda)(2 - \lambda) + 1 = 0$$

$$4 - 4\lambda + \lambda^2 + 1 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$\lambda_1 = 2 + i \quad \lambda_2 = 2 - i$$

$$\begin{aligned}
& (A - \lambda_1 I)\vec{v} = 0 \\
& \begin{bmatrix} 2 - (2 + i) & -1 & 0 \\ 1 & 2 - (2 + i) & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
& x_1 - ix_2 = 0 \\
& x_1 = ix_2 = is \\
& \vec{v} = s \begin{bmatrix} i \\ 1 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
& (A - \lambda_2 I)\vec{v} = 0 \\
& \begin{bmatrix} 2 - (2 - i) & -1 & 0 \\ 1 & 2 - (2 - i) & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & i & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
& x_1 + ix_2 = 0 \\
& x_1 = -ix_2 = -is \\
& \vec{v} = s \begin{bmatrix} -i \\ 1 \end{bmatrix}
\end{aligned}$$

Question 7

Consider matrix A .

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Give conditions on a, b, c, d such that A has the following.

$$\begin{aligned}
& \det(A - \lambda I) = 0 \\
& \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = 0 \\
& (a - \lambda)(d - \lambda) - bc = 0 \\
& ad - a\lambda - d\lambda + \lambda^2 - bc = 0 \\
& \lambda^2 + (-a - d)\lambda + (ad - bc) = 0 \\
& D = B^2 - 4AC \\
& = (-a - d)^2 - 4(1)(ad - bc) \\
& = a^2 + 2ad + d^2 - 4ad + 4bc \\
& = a^2 + d^2 - 2ad + 4bc
\end{aligned}$$

(a) two distinct real eigenvalues

$$a^2 + d^2 - 2ad + 4bc > 0$$

(b) one real eigenvalue

$$a^2 + d^2 - 2ad + 4bc = 0$$

(c) no real eigenvalues

$$a^2 + d^2 - 2ad + 4bc < 0$$

Question 8

Compute the determinant using cofactor expansion along the first row and along the first column.

$$\begin{aligned}
 \begin{vmatrix} 1 & 0 & 2 \\ 4 & 1 & 1 \\ 0 & 1 & 3 \end{vmatrix} &= 1 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} - 0 + 2 \begin{vmatrix} 4 & 1 \\ 0 & 1 \end{vmatrix} \\
 &= 1(3 - 1) + 2(4 - 0) \\
 &= 2 + 8 = 10 \\
 &= 1 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} - 4 \begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} + 0 \\
 &= 1(3 - 1) - 4(0 - 2) \\
 &= 2 - (-8) = 10
 \end{aligned}$$

Question 9

Compute the determinant using cofactor expansion along any row or column that seems convenient.

$$\begin{aligned}
 \begin{vmatrix} \tan \theta & \sin \theta & \cos \theta \\ \cos \theta & 0 & -\sin \theta \\ \sin \theta & 0 & \cos \theta \end{vmatrix} &= -\sin \theta \begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} + 0 - 0 \\
 &= -\sin \theta (\cos^2 \theta + \sin^2 \theta) \\
 &= -\sin \theta
 \end{aligned}$$

Question 10

Compute the determinant using cofactor expansion along any row or column that seems convenient.

$$\begin{aligned}
 \begin{vmatrix} 1 & -1 & 0 & 4 \\ 2 & 5 & 2 & 7 \\ 0 & 1 & 0 & 0 \\ 1 & 6 & 2 & 1 \end{vmatrix} &= 0 - 1 \begin{vmatrix} 1 & 0 & 4 \\ 2 & 2 & 7 \\ 1 & 2 & 1 \end{vmatrix} + 0 - 0 \\
 &= - \left(0 + 2 \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 1 & 4 \\ 2 & 7 \end{vmatrix} \right) \\
 &= -(2(1 - 4) - 2(7 - 8)) \\
 &= -(-6 + 2) \\
 &= 4
 \end{aligned}$$

Question 11

Compute the determinant using cofactor expansion along any row or column that seems convenient.

$$\begin{aligned}
 \begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & c \\ 0 & d & e & f \\ g & h & i & j \end{vmatrix} &= 0 - 0 + 0 - a \begin{vmatrix} 0 & 0 & b \\ 0 & d & e \\ g & h & i \end{vmatrix} \\
 &= -a \left(0 - 0 + b \begin{vmatrix} 0 & d \\ g & h \end{vmatrix} \right) \\
 &= -a(b(0 - dg)) \\
 &= abdg
 \end{aligned}$$

Question 12

Let $A = [a_{ij}]$ be a square matrix.

- If A has a zero row (column), then $\det(A) = 0$.
- If B is obtained by interchanging two rows (columns) of A , then $\det(B) = -\det(A)$.
- If A has two identical rows (columns), then $\det(A) = 0$.
- If B is obtained by multiplying a row (column) of A by k , then $\det(B) = k \det(A)$.
- If A, B, C are identical except that the i th row (column) of C is the sum of the i th rows (columns) of A and B , then $\det(C) = \det(A) + \det(B)$.
- If B is obtained by adding a multiple of one row (column) of A to another row (column), then $\det(B) = \det(A)$.

Evaluate the given determinant using elementary row and/or column operations and the theorem above to reduce the matrix to row echelon form.

$$\begin{aligned} A &= \begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 4 \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & -14 \\ 0 & 1 & 4 \end{vmatrix} \quad (R_2 - 5R_1)(f) \\ &= \begin{vmatrix} 1 & 0 & 3 \\ 0 & 1 & -14 \\ 0 & 0 & 18 \end{vmatrix} \quad (R_3 - R_2)(f) \\ \det(A) &= 1(1(18)) = 18 \end{aligned}$$

Question 13

Let $A = [a_{ij}]$ be a square matrix.

- If A has a zero row (column), then $\det(A) = 0$.
- If B is obtained by interchanging two rows (columns) of A , then $\det(B) = -\det(A)$.
- If A has two identical rows (columns), then $\det(A) = 0$.
- If B is obtained by multiplying a row (column) of A by k , then $\det(B) = k \det(A)$.
- If A, B, C are identical except that the i th row (column) of C is the sum of the i th rows (columns) of A and B , then $\det(C) = \det(A) + \det(B)$.
- If B is obtained by adding a multiple of one row (column) of A to another row (column), then $\det(B) = \det(A)$.

Evaluate the given determinant using elementary row and/or column operations and the theorem above to reduce the matrix to row echelon form.

$$\begin{aligned}
 A &= \begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & 6 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{vmatrix} \\
 &= \begin{vmatrix} 1 & 0 & 0 & 3 \\ 2 & 0 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 \end{vmatrix} && (R_1 + R_3, R_2 - 6R_3, R_4 - 4R_3)(f) \\
 &= \begin{vmatrix} 1 & 0 & 0 & 3 \\ 2 & 0 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 \end{vmatrix} && (R_4 - R_1)(f) \\
 &= \begin{vmatrix} 1 & 0 & 0 & 3 \\ 2 & 0 & 0 & 8 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 \end{vmatrix} && (R_2 - R_4)(f) \\
 &= \begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 \end{vmatrix} && (R_2 - 2R_1)(f) \\
 &= \begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 2 \end{vmatrix} && (R_2 \leftrightarrow R_3, R_3 \leftrightarrow R_4)(b) \\
 &= 1(1(1(-1(-1(4)))))) = 4
 \end{aligned}$$

Question 14

A square matrix A is invertible if and only if $\det(A) \neq 0$. Use the theorem to find all values of k for which A is invertible.

$$A = \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix}$$

$$\det(A) \neq 0$$

$$\begin{aligned}
 &= 0 + (k+1) \begin{vmatrix} k & 3 \\ k & k-1 \end{vmatrix} - 1 \begin{vmatrix} k & -k \\ k & -8 \end{vmatrix} \\
 &= (k+1)(k(k-1) - 3k) - 1(-8k + k^2) \\
 &= (k+1)(k^2 - k - 3k) + 8k - k^2 \\
 &= k^3 - 4k^2 + k^2 - 4k + 8k - k^2 \\
 &= k^3 - 4k^2 + 4k \\
 &= k(k^2 - 4k + 4) \\
 &= k(k-2)(k-2)
 \end{aligned}$$

$$k \neq 0, 2$$

Question 15

Assume that A and B are $n \times n$ matrices with $\det(A) = 3$ and $\det(B) = -4$. Find the indicated determinant.

$$\det(AB) = \det(A) \det(B) = -12$$

Question 16

Assume that A and B are $n \times n$ matrices with $\det(A) = 5$ and $\det(B) = -2$. Find the indicated determinant.

$$\det(A^2) = \det(AA) = \det(A) \det(A) = 25$$

Question 17

Assume that A and B are $n \times n$ matrices with $\det(A) = 5$ and $\det(B) = -4$. Find the indicated determinant.

$$\det(B^{-1}A) = \frac{1}{\det(B)} \det(A) = \frac{1}{-4} 5 = -\frac{5}{4}$$

Question 18

Assume that A and B are $n \times n$ matrices with $\det(A) = 3$ and $\det(B) = -2$. Find the indicated determinant.

$$\det(5B^T) = 5^n \det(B^T) = 5^n(-2)$$

Question 19

A and B are $n \times n$ matrices. If A is idempotent (that is, $A^2 = A$), find all possible values of $\det(A)$.

$$\begin{aligned} A^2 &= A \\ \det(A^2) &= \det(A) \\ \det(AA) &= \det(A) \\ \det(A)^2 &= \det(A) \\ \det(A)^2 - \det(A) &= 0 \\ \det(A)(\det(A) - 1) &= 0 \\ \det(A) &= 0, 1 \end{aligned}$$

Question 20

A and B are $n \times n$ matrices. A square matrix A is called **nilpotent** if $A^m = 0$ for some $m > 1$. Find all possible values of $\det(A)$ if A is nilpotent.

$$\begin{aligned} A^m &= 0 \\ \det(A^m) &= 0 \\ \det(AA^{m-1}) &= 0 \\ \det(A) \det(A^{m-1}) &= 0 \\ \det(A) &= 0 \end{aligned}$$

No other possible values can arise.

Question 21

Consider the following.

$$A = \begin{bmatrix} 1 & 5 \\ -2 & 8 \end{bmatrix}$$

(a) Compute the characteristic polynomial of A .

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1 - \lambda & 5 \\ -2 & 8 - \lambda \end{vmatrix} \\ &= (1 - \lambda)(8 - \lambda) - (-10) \\ &= 8 - 9\lambda + \lambda^2 + 10 \\ &= \lambda^2 - 9\lambda + 18 \end{aligned}$$

(b) Compute the eigenvalues and bases of the corresponding eigenspaces in A .

$$\begin{aligned} \lambda^2 - 9\lambda + 18 &= 0 \\ (\lambda - 6)(\lambda - 3) &= 0 \\ \lambda_1 &= 6 \quad \lambda_2 = 3 \\ (A - 6I)\vec{v}_1 &= 0 \\ \begin{bmatrix} -5 & 5 \\ -2 & 3 \end{bmatrix} \vec{v}_1 &= 0 \\ \begin{bmatrix} -5 & 5 & 0 \\ -2 & 2 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ x_1 &= x_2 = s \\ \vec{v}_1 &= s \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ (A - 3I)\vec{v}_2 &= 0 \\ \begin{bmatrix} -2 & 5 \\ -2 & 5 \end{bmatrix} \vec{v}_2 &= 0 \\ \begin{bmatrix} -2 & 5 & 0 \\ -2 & 5 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & -\frac{5}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ x_1 &= \frac{5}{2}x_2 = \frac{5}{2}s \\ \vec{v}_2 &= s \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix} \end{aligned}$$

Question 22

Consider the following.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 5 \end{bmatrix}$$

(a) Compute the characteristic polynomial of A .

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 1 - \lambda & 1 & 0 \\ 0 & -4 - \lambda & 1 \\ 0 & 0 & 5 - \lambda \end{vmatrix} \\ &= (1 - \lambda)(-4 - \lambda)(5 - \lambda) \end{aligned}$$

(b) Compute the eigenvalues and bases of the corresponding eigenspaces of A .

$$\lambda_1 = 1$$

$$(A - 1I)\vec{v}_1 = 0$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = s \quad x_2 = 0 \quad x_3 = 0$$

$$\vec{v}_1 = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_2 = -4$$

$$(A + 4I)\vec{v}_2 = 0$$

$$\begin{bmatrix} 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 9 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -\frac{1}{5}x_2 = -\frac{1}{5}s \quad x_3 = 0$$

$$\vec{v}_2 = s \begin{bmatrix} -\frac{1}{5} \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda_3 = 5$$

$$(A - 5I)\vec{v}_3 = 0$$

$$\begin{bmatrix} -4 & 1 & 0 & 0 \\ 0 & -9 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{36} & 0 \\ 0 & 1 & -\frac{1}{9} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{1}{36}x_3 \quad x_2 = \frac{1}{9}x_3 \quad x_3 = s$$

$$\vec{v}_3 = s \begin{bmatrix} \frac{1}{36} \\ \frac{1}{9} \\ 1 \end{bmatrix}$$

Question 23

Consider the following.

$$A = \begin{bmatrix} -3 & 18 & 0 \\ -1 & 7 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) Compute the characteristic polynomial of A .

$$\begin{aligned}
 \det(A - \lambda I) &= \begin{vmatrix} -3 - \lambda & 18 & 0 \\ -1 & 7 - \lambda & 1 \\ 0 & 1 & 1 - \lambda \end{vmatrix} \\
 &= 0 - 1 \begin{vmatrix} -3 - \lambda & 0 \\ -1 & 1 \end{vmatrix} + (1 - \lambda) \begin{vmatrix} -3 - \lambda & 18 \\ -1 & 7 - \lambda \end{vmatrix} \\
 &= -1(-3 - \lambda) + (1 - \lambda)((-3 - \lambda)(7 - \lambda) + 18) \\
 &= 3 + \lambda + (1 - \lambda)(-21 - 4\lambda + \lambda^2 + 18) \\
 &= 3 + \lambda + (1 - \lambda)(\lambda^2 - 4\lambda - 3) \\
 &= 3 + \lambda + (\lambda^2 - 4\lambda - 3 - \lambda^3 + 4\lambda^2 + 3\lambda) \\
 &= -\lambda^3 + 5\lambda^2 \\
 &= -\lambda^2(\lambda - 5)
 \end{aligned}$$

(b) Compute the eigenvalues and bases of the corresponding eigenspaces of A .

$$\begin{aligned}
 \lambda_1 &= \lambda_2 = 0 \\
 (A - 0)v_1 &= 0 \\
 \begin{bmatrix} -3 & 18 & 0 & 0 \\ -1 & 7 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 x_1 &= -6x_3 \quad x_2 = -x_3 \quad x_3 = s \\
 v_1 = v_2 &= s \begin{bmatrix} -6 \\ -1 \\ 1 \end{bmatrix} \\
 (A - 5I)v_3 &= 0 \\
 \begin{bmatrix} -8 & 18 & 0 & 0 \\ -1 & 2 & 1 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & -9 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
 x_1 &= 9x_3 \quad x_2 = 4x_3 \quad x_3 = s \\
 v_3 &= s \begin{bmatrix} 9 \\ 4 \\ 1 \end{bmatrix}
 \end{aligned}$$

Question 24

Let $p(x)$ be the polynomial

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

The **companion matrix** of $p(x)$ is the $n \times n$ matrix

$$C(p) = \begin{bmatrix} -a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

Find the companion matrix of $p(x) = x^3 + 2x^2 - 4x + 10$ and then find the characteristic polynomial of $C(p)$.

$$\begin{aligned}
 C(p) &= \begin{bmatrix} -2 & 4 & -10 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\
 \det(C(p) - \lambda I) &= \begin{vmatrix} -2-\lambda & 4 & -10 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} \\
 &= 0 - 1 \begin{vmatrix} -2-\lambda & -10 \\ 1 & 0 \end{vmatrix} - \lambda \begin{vmatrix} -2-\lambda & 4 \\ 1 & -\lambda \end{vmatrix} \\
 &= -1(0 + 10) - \lambda(2\lambda + \lambda^2 - 4) \\
 &= -10 - 2\lambda^2 - \lambda^3 + 4\lambda
 \end{aligned}$$

Question 25

For the matrix A , use the Cayley-Hamilton Theorem to express A^3 and A^4 as a linear combination of I, A, A^2 .

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \\
 c_A(\lambda) &= \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} \\
 &= (1-\lambda) \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ 0 & 1-\lambda \end{vmatrix} + 0 \\
 &= (1-\lambda)(-\lambda + \lambda^2 - 1) - 1(1-\lambda) \\
 &= -\lambda + \lambda^2 - 1 + \lambda^2 - \lambda^3 + \lambda - 1 + \lambda \\
 &= -\lambda^3 + 2\lambda^2 + \lambda - 2 \\
 -A^3 + 2A^2 + A - 2I &= 0 \\
 A^3 &= 2A^2 + A - 2I \\
 A^4 &= AA^3 \\
 &= A(2A^2 + A - 2I) \\
 &= 2A^3 + A^2 - 2A \\
 &= 2(2A^2 + A - 2I) + A^2 - 2A \\
 &= 4A^2 + 2A - 4I + A^2 - 2A \\
 &= 5A^2 - 4I
 \end{aligned}$$

Question 26

Show that A and B are not similar matrices.

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 1 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}$$
$$c_A(\lambda) = \begin{vmatrix} 1-\lambda & 5 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix}$$
$$= (1-\lambda) \begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - 0 + 0$$
$$= (1-\lambda)((1-\lambda)^2 - 1)$$
$$= (1-\lambda)(1-2\lambda+\lambda^2-1)$$
$$= \lambda(1-\lambda)(\lambda-2)$$
$$c_B(\lambda) = \begin{vmatrix} 5-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 5 & 0 & 1-\lambda \end{vmatrix}$$
$$= (5-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 0 + 5 \begin{vmatrix} 1 & 1 \\ 1-\lambda & 0 \end{vmatrix}$$
$$= (5-\lambda)(1-2\lambda+\lambda^2) + 5(0 - (1-\lambda))$$
$$= 5 - 10\lambda + 5\lambda^2 - \lambda + 2\lambda^2 - \lambda^3 - 5 + 5\lambda$$
$$= -\lambda^3 + 7\lambda^2 - 6\lambda$$
$$= -\lambda(\lambda-6)(\lambda-1)$$

Question 27

A diagonalization of the matrix A is given in the form $P^{-1}AP = D$. List the eigenvalues of A and bases for the corresponding eigenspaces.

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$$
$$\lambda_1 = 4 \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\lambda_2 = 5 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Question 28

A diagonalization of the matrix A is given in the form $P^{-1}AP = D$. List the eigenvalues of A and bases for the corresponding eigenspaces.

$$\begin{bmatrix} \frac{1}{14} & \frac{1}{14} & \frac{1}{14} \\ \frac{1}{7} & \frac{1}{7} & \frac{1}{7} \\ \frac{3}{7} & \frac{3}{7} & \frac{3}{7} \end{bmatrix} \begin{bmatrix} 3 & 3 & 3 \\ 0 & 0 & 1 \\ 3 & 3 & 2 \end{bmatrix} \begin{bmatrix} 7 & 1 & 0 \\ 1 & -1 & 0 \\ 6 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
$$\lambda_1 = 6 \quad \vec{v}_1 = \begin{bmatrix} 7 \\ 1 \\ 6 \end{bmatrix}$$
$$\lambda_2 = 0 \quad \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$
$$\lambda_3 = -1 \quad \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Question 29

Determine whether A is diagonalizable.

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$\begin{aligned} c_A(\lambda) &= \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{vmatrix} \\ &= 1 \begin{vmatrix} 1 & 1 \\ 1-\lambda & 0 \end{vmatrix} - 0 + (1-\lambda) \begin{vmatrix} -\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} \\ &= -(1-\lambda) + (1-\lambda)(-\lambda + \lambda^2 - 1) \\ &= -1 + \lambda - \lambda + \lambda^2 - 1 + \lambda^2 - \lambda^3 + \lambda \\ &= -\lambda^3 + 2\lambda^2 + \lambda - 2 \\ &= (\lambda + 1)(\lambda - 1)(\lambda - 2) \end{aligned}$$

$$\lambda_1 = -1$$

$$(A + 1I)\vec{v}_1 = 0$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -2x_3 \quad x_2 = x_3 \quad x_3 = s$$

$$\vec{v}_1 = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1$$

$$(A - 1I)\vec{v}_2 = 0$$

$$\begin{bmatrix} -1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 0 \quad x_2 = -x_3 \quad x_3 = s$$

$$\vec{v}_2 = s \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 2$$

$$(A - 2I)\vec{v}_3 = 0$$

$$\begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_3 \quad x_2 = x_3 \quad x_3 = s$$

$$\vec{v}_3 = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P = \begin{bmatrix} -2 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Question 30

Determine whether A is diagonalizable.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 4 & 1 \\ 5 & 0 & 1 \end{bmatrix}$$

Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

$$\begin{aligned} c_A(\lambda) &= \begin{vmatrix} 1-\lambda & 0 & 0 \\ 4 & 4-\lambda & 1 \\ 5 & 0 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda) \begin{vmatrix} 4-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} - 0 + 0 \\ &= (1-\lambda)(4-\lambda)(1-\lambda) \end{aligned}$$

$$\lambda_1 = \lambda_2 = 1$$

$$(A - 1I)v_{12} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 4 & 3 & 1 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 0 \quad x_2 = \frac{1}{3}x_3 \quad x_3 = 0$$

$$v_{12} = s \begin{bmatrix} 0 \\ \frac{1}{3} \\ 1 \end{bmatrix}$$

The geometric multiplicity of this eigenvalue is less than its algebraic multiplicity, so this matrix is not diagonalizable.

Question 31

Use the method of Example 4.29 to compute the indicated power of the matrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{2017}$$

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\ c_A(\lambda) &= \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} \\ &= (1-\lambda) \begin{vmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} - 0 + 0 \\ &= (1-\lambda)(-1-\lambda)(-1-\lambda) \end{aligned}$$

$$\lambda_1 = 1$$

$$(A - 1I)v_1 = 0$$

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$v_1 = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda_{23} = -1$$

$$(A + 1I)v_{23} = 0$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$2x_1 = -x_2 - x_3 \quad x_2 = s \quad x_3 = t$$

$$v_{23} = s \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^{-1}AP = D$$

$$A = PDP^{-1}$$

$$A^{2017} = PD^{2017}P^{-1}$$

$$= \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}^{2017} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech