

Advanced Linear Algebra: Homework 1

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Question 1

Refer to the vectors below.

$$\vec{b} = [4, 2, 1] \quad \vec{c} = [1, -3, 1] \quad \vec{d} = [-1, -1, -2]$$

Compute the indicated vector.

$$\begin{aligned} 3\vec{b} - 2\vec{c} + \vec{d} &= [12, 6, 3] - [2, -6, 2] + [-1, -1, -2] \\ &= [9, 11, -1] \end{aligned}$$

Question 2

$$\begin{aligned} \vec{u} &= \left[\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right) \right] \\ \vec{v} &= \left[-\cos\left(\frac{\pi}{6}\right), -\sin\left(\frac{\pi}{6}\right) \right] \\ \vec{u} + \vec{v} &= \left[\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{6}\right) \right] \\ \vec{u} - \vec{v} &= \left[\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right), \sin\left(\frac{\pi}{3}\right) + \sin\left(\frac{\pi}{6}\right) \right] \end{aligned}$$

Question 3

Find the projection of \vec{v} onto \vec{u} .

$$\begin{aligned} \vec{u} &= \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{2} \end{bmatrix} & \vec{v} &= \begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix} \\ \text{proj}_{\vec{u}}(\vec{v}) &= \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}} \vec{u} \\ &= \frac{2 - 1 + 2}{\frac{1}{4} + \frac{1}{16} + \frac{1}{4}} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{2} \end{bmatrix} \\ &= \frac{3}{\frac{9}{16}} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{8}{3} \\ 4 \\ -3 \end{bmatrix} \end{aligned}$$

Question 4

Find $\vec{u} \cdot \vec{v}$.

$$\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = (3)(3) + (1)(2) + (2)(1) = 13$$

Question 5

Find $\|\vec{u}\|$ for the given vector.

$$\vec{u} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

$$\|\vec{u}\| = \sqrt{(3)(3) + (4)(4) + (1)(1)} = \sqrt{26}$$

Give a unit vector in the direction of \vec{u} .

$$\frac{\vec{u}}{\|\vec{u}\|} = \begin{bmatrix} \frac{3}{\sqrt{26}} \\ \frac{4}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} \end{bmatrix}$$

Question 6

Find the angle between \vec{u} and \vec{v} .

$$\vec{u} = [4, 3, -1] \quad \vec{v} = [1, -1, 1]$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\theta = \cos^{-1}\left(\frac{4 - 3 - 1}{(16 + 9 + 1)(1 + 1 + 1)}\right)$$

$$= \cos^{-1}(0) = \frac{\pi}{2}$$

Question 7

Find the angle between \vec{u} and \vec{v} .

$$\vec{u} = [0.9, 1.9, 1.2] \quad \vec{v} = [-4.5, 2.4, -0.8]$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\theta = \cos^{-1}\left(\frac{-4.05 + 4.56 - 0.96}{\sqrt{0.81 + 3.61 + 1.44} \sqrt{20.25 + 5.76 + 0.64}}\right)$$

$$= \cos^{-1}\left(\frac{-0.45}{(2.42)(5.16)}\right) \approx 92.06^\circ$$

Question 8

Find all values of the scalar k for which the two vectors are orthogonal.

$$\vec{u} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$\vec{v} = \begin{bmatrix} k^2 \\ k \\ -4 \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = 0$$

$$k^2 - k - 12 = 0$$

$$(k - 4)(k + 3) = 0$$

$$k = 4, -3$$

Question 9

Write the equation of the line passing through P with direction vector \vec{d} in vector form and parametric form.

$$P = (5, 0, -3) \quad \vec{d} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$x = 5 + 3t$$

$$y = 2t$$

$$z = -3$$

Question 10

Write the equation of the plane passing through P with normal vector \vec{n} in normal form and general form.

$$P = (0, 1, 0) \quad \vec{n} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = 0$$

$$3x + 4(y - 1) + z = 0$$

$$3x + 4y + z = -4$$

Question 11

Write the equation of the plane passing through P with direction vectors \vec{u} and \vec{v} in vector form and parametric form.

$$P = (0, 0, 0) \quad \vec{u} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

$$x = 3s - 4t$$

$$y = s + 3t$$

$$z = 3s + t$$

Question 12

Given the vector equation of the line passing through P and Q .

$$P = (0, 1, -1) \quad (-4, 1, 6)$$

$$P - Q = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}$$

Question 13

Find the normal form of the equation of the plane that passes through $P = (0, -2, 5)$ and is parallel to the plane with general equation $4x - y + 5z = 3$.

$$\vec{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$$

$$\vec{n} \cdot \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} = 27$$

$$\begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 27$$

Question 14

Solve the given system by back substitution.

$$\begin{aligned}x - 3y + z &= 5 \\y - 2z &= -1 \\y &= -1 + 2z \\x - 3(-1 + 2z) + z &= 5 \\x + 3 - 6z + z &= 5 \\x &= 2 + 5z \\z &= s \\x &= 2 + 5s \\y &= -1 + 2s\end{aligned}$$

Question 15

Find a system of linear equations that has the given matrix as its augmented matrix.

$$\left[\begin{array}{cccc} 0 & 1 & 1 & 1 \\ 1 & -1 & 0 & 1 \\ 4 & -1 & 1 & 1 \end{array} \right]$$

$$\begin{aligned}y + z &= 1 \\x - y &= 1 \\4x - y + z &= 1\end{aligned}$$

Question 16

Solve the linear system.

$$\begin{aligned}-2x_1 + 3x_2 - x_3 &= 1 \\x_1 + x_3 &= 0 \\-x_1 + 2x_2 - 2x_3 &= 0\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} -2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & -2 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} -2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & \frac{5}{2} & 1 \\ 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{2}{5} \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & 0 & -\frac{2}{5} \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{5} \end{bmatrix} \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &= \begin{bmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ \frac{2}{5} \end{bmatrix}
\end{aligned}$$

Question 17

Solve the linear system.

$$\begin{aligned}
a - 2b + d &= 3 \\
-a + b - c - 4d &= 1
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix} 1 & -2 & 0 & 1 & 3 \\ -1 & 1 & -1 & -4 & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & -2 & 0 & 1 & 3 \\ 0 & -1 & -1 & -3 & 4 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & 2 & 7 & -5 \\ 0 & -1 & -1 & -3 & 4 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & 2 & 7 & -5 \\ 0 & 1 & 1 & 3 & -4 \end{bmatrix}
\end{aligned}$$

$$c = s \quad d = t$$

$$a + 2c + 7d = -5$$

$$a = -5 - 2c - 7d$$

$$b + c + 3d = -4$$

$$b = -4 - c - 3d$$

Question 18

Determine whether the given matrix is in row echelon form.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

The matrix is not in row echelon form.

Question 19

Determine whether the given matrix is in row echelon form.

$$\begin{bmatrix} 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix is in row echelon form and reduced row echelon form.

Question 20

Use elementary row operations to reduce the given matrix to row echelon form and reduced row echelon form.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
$$REF = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$
$$RREF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Question 21

Use elementary row operations to reduce the given matrix to row echelon form and reduced row echelon form.

$$\begin{bmatrix} -2 & -4 & 9 \\ -4 & -8 & 17 \\ 1 & 2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -4 & 9 \\ -4 & -8 & 17 \\ 1 & 2 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ -4 & -8 & 17 \\ 1 & 2 & -4 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & -4 \end{bmatrix}$$
$$\rightarrow \begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} (REF)$$
$$\rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} (RREF)$$

Question 22

What is the rank of each of the matrices below?

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix} 3$$

$$\begin{bmatrix} 8 & 0 & 1 & 0 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} 2$$

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} 2$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} 0$$

$$\begin{bmatrix} 1 & 0 & 4 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 \end{bmatrix} 2$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} 3$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} 3$$

$$\begin{bmatrix} 2 & 1 & 4 & 6 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} 3$$

Question 23

Solve the given system of equations using either Gaussian or Gauss-Jordan elimination.

$$x_1 + 2x_2 - 3x_3 = 13$$

$$3x_1 - x_2 + x_3 = 0$$

$$4x_1 - x_2 + x_3 = 2$$

$$\begin{aligned}
\begin{bmatrix} 1 & 2 & -3 & 13 \\ 3 & -1 & 1 & 0 \\ 4 & -1 & 1 & 2 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & -7 & 10 & -39 \\ 4 & -1 & 1 & 2 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & -7 & 10 & -39 \\ 0 & -9 & 13 & -50 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & 1 & \frac{-10}{7} & \frac{39}{7} \\ 0 & -9 & 13 & -50 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & 1 & \frac{-10}{7} & \frac{39}{7} \\ 0 & 0 & \frac{7}{7} & \frac{7}{7} \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & 1 & \frac{-10}{7} & \frac{39}{7} \\ 0 & 0 & 1 & 1 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & 1 & 0 & \frac{49}{7} \\ 0 & 0 & 1 & 1 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix}
\end{aligned}$$

Question 24

Solve the given system of equations using either Gaussian or Gauss-Jordan elimination.

$$\begin{aligned}
-x_1 + 4x_2 - 2x_3 + 4x_4 &= 0 \\
2x_1 - 8x_2 + x_3 - 2x_4 &= -3 \\
x_1 - 4x_2 + 4x_3 - 8x_4 &= 2
\end{aligned}$$

$$\begin{bmatrix} -1 & 4 & -2 & 4 & 0 \\ 2 & -8 & 1 & -2 & -3 \\ 1 & -4 & 4 & -8 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -4 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}
x_1 - 4x_2 &= -2 \\
x_3 - 2x_4 &= 1 \\
x_2 = s \quad x_4 &= t \\
x_1 &= -2 + 4s \\
x_3 &= 1 + 2t
\end{aligned}$$

Question 25

Solve the given system of equations using either Gaussian or Gauss-Jordan elimination.

$$\begin{aligned}\frac{1}{2}x_1 + x_2 - x_3 - 6x_4 &= 2 \\ \frac{1}{6}x_1 + \frac{1}{2}x_2 - 3x_4 + x_5 &= -1 \\ \frac{1}{3}x_1 - 2x_3 - 4x_5 &= 8\end{aligned}$$

$$\begin{bmatrix} \frac{1}{2} & 1 & -1 & -6 & 0 & 2 \\ \frac{1}{6} & \frac{1}{2} & 0 & -3 & 1 & -1 \\ \frac{1}{3} & 0 & -2 & 0 & -4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -6 & 0 & -12 & 24 \\ 0 & 1 & 2 & -6 & 6 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned}x_1 - 6x_3 - 12x_5 &= 24 \\ x_2 + 2x_3 - 6x_4 + 6x_5 &= -10 \\ x_3 = r \quad x_4 = s \quad x_5 = t \\ x_1 &= 24 + 6r + 12t \\ x_2 &= -10 - 2r + 6s - 6t\end{aligned}$$

Question 26

Solve the given system of equations using either Gaussian or Gauss-Jordan elimination.

$$\begin{aligned}w + x + 2y + z &= 1 \\ w - x - y + z &= 0 \\ x + y &= -1 \\ w + x + z &= 4\end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 4 \end{bmatrix}$$

No solution.

Question 27

Use elementary row operations to reduce the given matrix to row echelon form and reduced row echelon form.

$$\begin{aligned} \begin{bmatrix} -2 & 6 & -7 \\ 3 & -9 & 10 \\ -6 & 18 & -19 \end{bmatrix} &\rightarrow \begin{bmatrix} -2 & 6 & -7 \\ 3 & -9 & 10 \\ 0 & 0 & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} -1 & 3 & -\frac{7}{2} \\ 3 & -9 & 10 \\ 0 & 0 & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} -1 & 3 & -\frac{7}{2} \\ 0 & 0 & -\frac{7}{2} \\ 0 & 0 & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & -3 & \frac{7}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ (REF)} \\ &\rightarrow \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ (RREF)} \end{aligned}$$

Question 28

Determine if the vector \vec{v} is a linear combination of the remaining vectors.

$$\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix} \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$x\vec{u}_1 + y\vec{u}_2 + z\vec{u}_3 = \vec{v}$$

$$x + z = 4$$

$$x + y = 3$$

$$x + y + z = 4$$

$$\begin{bmatrix} 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\therefore 3\vec{u}_1 + \vec{u}_3 = \vec{v}$$

Question 29

Determine if the vector \vec{b} is in the span of the columns of the matrix A .

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix}$$
$$x \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} + z \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix}$$
$$\begin{aligned} x + 2y + 3z &= 4 \\ 4x + 5y + 6z &= 7 \\ 6x + 7y + 8z &= 9 \end{aligned}$$
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This system has a solution so \vec{b} can be represented as a linear combination of the columns of A , therefore it is in the span.

Question 30

Describe the span of the given vectors geometrically and algebraically.

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$$

The set of all linear combinations of \vec{u} and \vec{v} is a plane containing both \vec{u} and \vec{v} as direction vectors and contains all points described by $s\vec{u} + t\vec{v}$ where s, t are arbitrary constants. The vector equation of this plane is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$$

Question 31

Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ be (column) vectors in \mathbb{R}^n and let A be the $n \times m$ matrix $[\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m]$ with these vectors as its columns. Then $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ are linearly dependent if and only if the homogeneous linear system with augmented matrix $[A][0]$ has a nontrivial solution. Use the theorem above to determine if the set of vectors is linearly dependent.

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 2 & 1 & -5 & 0 \\ 1 & 2 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The set of vectors is linearly dependent.

Question 32

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible. Find the inverse of the given matrix (if it exists) using the theorem above.

$$\begin{bmatrix} 4 & 19 \\ 1 & 5 \end{bmatrix}$$

$$ad - bc = 20 - 19 = 1$$

$$\begin{aligned} \begin{bmatrix} 4 & 19 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{bmatrix} &\rightarrow \begin{bmatrix} 0 & -1 & 1 & -4 \\ 1 & 5 & 0 & 1 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 0 & -1 & 1 & -4 \\ 1 & 0 & 5 & -19 \end{bmatrix} \\ &\rightarrow \begin{bmatrix} 1 & 0 & 5 & -19 \\ 0 & 1 & -1 & 4 \end{bmatrix} \end{aligned}$$

Question 33

Use the Gauss-Jordan method to find the inverse of the given matrix (if it exists).

$$\begin{bmatrix} 3 & 2 & 0 \\ -2 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} 3 & 2 & 0 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 3 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 6 & 3 & -3 & 0 & 0 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 3 & 2 & 0 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 0 & 0 & -6 & -12 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 & -1 & -2 & 0 \\ 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix} \end{aligned}$$

Question 34

Determine whether \vec{b} is in $\text{col}(A)$. Determine whether \vec{w} is in $\text{row}(A)$.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \vec{w} = [-1 \ 1 \ 1]$$

$$\begin{bmatrix} 1 & 0 & -1 & 4 \\ 1 & 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

This augmented matrix has a solution, so \vec{b} is a linear combination of the columns of A , therefore it is in the column space of A .

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

This system does not have a solution, so \vec{w} is not a linear combination of the rows of A , therefore it is not in the row space of A .

Question 35

Determine whether \vec{b} is in $\text{col}(A)$. Determine whether \vec{w} is in $\text{row}(A)$.

$$A = \begin{bmatrix} 1 & 1 & -5 \\ 0 & 2 & 1 \\ 1 & -1 & 6 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = [2 \ 4 \ -9]$$

$$\begin{bmatrix} 1 & 1 & -5 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This augmented matrix has a solution, so \vec{b} is a linear combination of the columns of A , therefore it is in the column space of A .

$$\begin{bmatrix} -5 & 1 & 6 & -9 \\ 1 & 2 & -1 & 4 \\ 1 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This augmented matrix has a solution, so \vec{w} is a linear combination of the rows of A , therefore it is in the row space of A .

Question 36

If $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, is $\vec{v} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$ in $\text{null}(A)$?

$$\begin{aligned} A\vec{v} &= \begin{bmatrix} (1)(-1) + (0)(4) + (-1)(-1) \\ (1)(-1) + (1)(4) + (1)(-1) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{aligned}$$

\vec{v} is not in the null space of A .

Question 37

Give bases for $\text{row}(A)$, $\text{col}(A)$, $\text{null}(A)$.

$$A = \begin{bmatrix} 2 & -4 & 2 & 1 & 0 \\ -1 & 2 & 1 & 3 & 2 \\ 1 & -2 & 3 & 4 & 2 \end{bmatrix}$$

$$\text{RREF}(A) = \begin{bmatrix} 1 & -2 & 0 & -\frac{5}{4} & -1 \\ 0 & 0 & 1 & \frac{7}{4} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{col}(A) = \text{span} \left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right)$$

$$x_1 - 2x_2 - \frac{5}{4}x_4 - x_5 = 0$$

$$x_3 + \frac{7}{4}x_4 + x_5 = 0$$

$$x_2 = r \quad x_4 = s \quad x_5 = t$$

$$x_1 = 2r + \frac{5}{4}s + t$$

$$x_3 = -\frac{7}{4}s - t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{null}(A) = \text{span} \left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Question 38

Find a basis \mathbb{B} for the span of the given vectors.

$$\begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{RREF} \left(\begin{bmatrix} 4 & -5 & 1 \\ 1 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbb{B} = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Question 39

Give the rank and nullity of the matrix.

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix}$$
$$RREF(A) = \begin{bmatrix} 1 & 0 & -\frac{7}{2} \\ 0 & 2 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$
$$rank(A) = 2$$
$$nullity(A) = 1$$

Question 40

Give the rank and nullity of the matrix below.

$$A = \begin{bmatrix} 2 & -5 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 4 \\ 1 & -3 & 1 & 4 & 5 \end{bmatrix}$$
$$RREF(A) = \begin{bmatrix} 1 & 0 & -5 & -14 & -22 \\ 0 & 1 & -2 & -6 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$rank(A) = 2$$
$$x_1 - 5x_3 - 14x_4 - 22x_5 = 0$$
$$x_2 - 2x_3 - 6x_4 - 9x_5 = 0$$
$$nullity(A) = 3 \quad (\text{free variables : } x_3, x_4, x_5)$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech