# Advanced Linear Algebra: Homework 1

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### Question 1

Refer to the vectors below.

$$\vec{b} = [4, 2, 1] \quad \vec{c} = [1, -3, 1] \quad \vec{d} = [-1, -1, -2]$$

Compute the indicated vector.

$$\begin{aligned} 3\vec{b} - 2\vec{c} + \vec{d} &= [12, 6, 3] - [2, -6, 2] + [-1, -1, -2] \\ &= [9, 11, -1] \end{aligned}$$

### Question 2

$$\begin{aligned} \vec{u} &= \left[\cos(\frac{\pi}{3}), \sin(\frac{\pi}{3})\right] \\ \vec{v} &= \left[-\cos(\frac{\pi}{6}), -\sin(\frac{\pi}{6})\right] \\ \vec{u} + \vec{v} &= \left[\cos(\frac{\pi}{3}) - \cos(\frac{\pi}{6}), \sin(\frac{\pi}{3}) - \sin(\frac{\pi}{6})\right] \\ \vec{u} - \vec{v} &= \left[\cos(\frac{\pi}{3}) + \cos(\frac{\pi}{6}), \sin(\frac{\pi}{3}) + \sin(\frac{\pi}{6})\right] \end{aligned}$$

### Question 3

Find the projection of $\vec{v}$ onto $\vec{u}$.

$$\vec{u} = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{2} \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 4 \\ 4 \\ -4 \end{bmatrix}$$

$$\begin{aligned} \text{proj}_{\vec{u}}(\vec{v}) &= \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\vec{u} \\ &= \frac{2 - 1 + 2}{\frac{1}{4} + \frac{1}{16} + \frac{1}{4}} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{2} \end{bmatrix} \\ &= \frac{3}{\frac{9}{16}} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{4} \\ -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{8}{3} \\ -\frac{4}{3} \\ -\frac{8}{3} \end{bmatrix} \end{aligned}$$

**Question 4**

Find $\vec{u} \cdot \vec{v}$.

$$\vec{u} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$
$$\vec{v} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$
$$\vec{u} \cdot \vec{v} = (3)(3) + (1)(2) + (2)(1) = 13$$

**Question 5**

Find $\|\vec{u}\|$ for the given vector.

$$\vec{u} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$
$$\|\vec{u}\| = \sqrt{(3)(3) + (4)(4) + (1)(1)} = \sqrt{26}$$

Give a unit vector in the direction of $\vec{u}$.

$$\frac{\vec{u}}{\|\vec{u}\|} = \begin{bmatrix} \frac{3}{\sqrt{26}} \\ \frac{4}{\sqrt{26}} \\ \frac{1}{\sqrt{26}} \end{bmatrix}$$

**Question 6**

Find the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = [4, 3, -1] \quad \vec{v} = [1, -1, 1]$$
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$
$$\theta = \cos^{-1}\left(\frac{4 - 3 - 1}{(16 + 9 + 1)(1 + 1 + 1)}\right)$$
$$= \cos^{-1}(0) = \frac{\pi}{2}$$

**Question 7**

Find the angle between $\vec{u}$ and $\vec{v}$.

$$\vec{u} = [0.9, 1.9, 1.2] \quad \vec{v} = [-4.5, 2.4, -0.8]$$
$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|}$$
$$\theta = \cos^{-1}\left(\frac{-4.05 + 4.56 - 0.96}{\sqrt{0.81 + 3.61 + 1.44}\sqrt{20.25 + 5.76 + 0.64}}\right)$$
$$= \cos^{-1}\left(\frac{-0.45}{(2.42)(5.16)}\right) \approx 92.06^\circ$$

### Question 8

Find all values of the scalar $k$ for which the two vectors are orthogonal.

$$\begin{aligned}\vec{u} &= \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \\ \vec{v} &= \begin{bmatrix} k^2 \\ k \\ -4 \end{bmatrix} \\ \vec{u} \cdot \vec{v} &= 0 \\ k^2 - k - 12 &= 0 \\ (k-4)(k+3) &= 0 \\ k &= 4, -3\end{aligned}$$

### Question 9

Write the equation of the line passing through $P$ with direction vector $\vec{d}$ in vector form and parametric form.

$$P = (5, 0, -3) \quad \vec{d} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \end{bmatrix} + t \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$\begin{aligned}x &= 5 + 3t \\ y &= 2t \\ z &= -3\end{aligned}$$

### Question 10

Write the equation of the plane passing through $P$ with normal vector $\vec{n}$ in normal form and general form.

$$P = (0, 1, 0) \quad \vec{n} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \cdot \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = 0$$

$$\begin{aligned}3x + 4(y-1) + z &= 0 \\ 3x + 4y + z &= -4\end{aligned}$$

## Question 11

Write the equation of the plane passing through $P$ with direction vectors $\vec{u}$ and $\vec{v}$ in vector form and parametric form.

$$P = (0,0,0) \quad \vec{u} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{aligned} x &= 3s - 4t \\ y &= s + 3t \\ z &= 3s + t \end{aligned}$$

## Question 12

Given the vector equation of the line passing through $P$ and $Q$.

$$P = (0, 1, -1) \quad (-4, 1, 6)$$

$$P - Q = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}$$

## Question 13

Find the normal form of the equation of the plane that passes through $P = (0, -2, 5)$ and is parallel to the plane with general equation $4x - y + 5z = 3$.

$$\vec{n} = \begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix}$$

$$\vec{n} \cdot \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} = 27$$

$$\begin{bmatrix} 4 \\ -1 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 27$$

**Question 14**

Solve the given system by back substitution.

$$
\begin{aligned}
x - 3y + z &= 5 \\
y - 2z &= -1 \\
y &= -1 + 2z \\
x - 3(-1 + 2z) + z &= 5 \\
x + 3 - 6z + z &= 5 \\
x &= 2 + 5z \\
z &= s \\
x &= 2 + 5s \\
y &= -1 + 2s
\end{aligned}
$$

**Question 15**

Find a system of linear equations that has the given matrix as its augmented matrix.

$$
\begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & -1 & 0 & 1 \\
4 & -1 & 1 & 1
\end{bmatrix}
$$

$$
\begin{aligned}
y + z &= 1 \\
x - y &= 1 \\
4x - y + z &= 1
\end{aligned}
$$

**Question 16**

Solve the linear system.

$$
\begin{aligned}
-2x_1 + 3x_2 - x_3 &= 1 \\
x_1 + x_3 &= 0 \\
-x_1 + 2x_2 - 2x_3 &= 0
\end{aligned}
$$

$$\begin{bmatrix} -2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ -1 & 2 & -2 & 0 \end{bmatrix} \to \begin{bmatrix} -2 & 3 & -1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 3 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & \frac{5}{2} & 1 \\ 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{2}{5} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -\frac{2}{5} \\ 0 & 1 & 0 & \frac{1}{5} \\ 0 & 0 & 1 & \frac{2}{5} \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ \frac{2}{5} \end{bmatrix}$$

### Question 17

Solve the linear system.

$$\begin{aligned} a - 2b + d &= 3 \\ -a + b - c - 4d &= 1 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 0 & 1 & 3 \\ -1 & 1 & -1 & -4 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 1 & 3 \\ 0 & -1 & -1 & -3 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & 7 & -5 \\ 0 & -1 & -1 & -3 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & 7 & -5 \\ 0 & 1 & 1 & 3 & -4 \end{bmatrix}$$

$$c = s \quad d = t$$

$$\begin{aligned} a + 2c + 7d &= -5 \\ a &= -5 - 2c - 7d \\ b + c + 3d &= -4 \\ b &= -4 - c - 3d \end{aligned}$$

### Question 18

Determine whether the given matrix is in row echelon form.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}$$

The matrix is not in row echelon form.

**Question 19**

Determine whether the given matrix is in row echelon form.

$$\begin{bmatrix} 0 & 1 & 4 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrix is in row echelon form and reduced row echelon form.

**Question 20**

Use elementary row operations to reduce the given matrix to row echelon form and reduced row echelon form.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$REF = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$RREF = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

**Question 21**

Use elementary row operations to reduce the given matrix to row echelon form and reduced row echelon form.

$$\begin{bmatrix} -2 & -4 & 9 \\ -4 & -8 & 17 \\ 1 & 2 & -4 \end{bmatrix}$$

$$\begin{aligned} \begin{bmatrix} -2 & -4 & 9 \\ -4 & -8 & 17 \\ 1 & 2 & -4 \end{bmatrix} &\to \begin{bmatrix} 0 & 0 & 1 \\ -4 & -8 & 17 \\ 1 & 2 & -4 \end{bmatrix} \\ &\to \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 2 & -4 \end{bmatrix} \\ &\to \begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} (REF) \\ &\to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} (RREF) \end{aligned}$$

## Question 22

What is the rank of each of the matrices below?

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix} 3$$

$$\begin{bmatrix} 8 & 0 & 1 & 0 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} 2$$

$$\begin{bmatrix} 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} 2$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} 0$$

$$\begin{bmatrix} 1 & 0 & 4 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 \end{bmatrix} 2$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} 3$$

$$\begin{bmatrix} 1 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} 3$$

$$\begin{bmatrix} 2 & 1 & 4 & 6 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} 3$$

## Question 23

Solve the given system of equations using either Gaussian or Gauss-Jordan elimination.

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 13 \\ 3x_1 - x_2 + x_3 &= 0 \\ 4x_1 - x_2 + x_3 &= 2 \end{aligned}$$

$$\begin{bmatrix} 1 & 2 & -3 & 13 \\ 3 & -1 & 1 & 0 \\ 4 & -1 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & -7 & 10 & -39 \\ 4 & -1 & 1 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & -7 & 10 & -39 \\ 0 & -9 & 13 & -50 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & 1 & \frac{-10}{7} & \frac{39}{7} \\ 0 & -9 & 13 & -50 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & 1 & \frac{-10}{7} & \frac{39}{7} \\ 0 & 0 & \frac{1}{7} & \frac{1}{7} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & 1 & \frac{-10}{7} & \frac{39}{7} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & 1 & 0 & \frac{49}{7} \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -3 & 13 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

### Question 24

Solve the given system of equations using either Gaussian or Gauss-Jordan elimination.

$$\begin{aligned} -x_1 + 4x_2 - 2x_3 + 4x_4 &= 0 \\ 2x_1 - 8x_2 + x_3 - 2x_4 &= -3 \\ x_1 - 4x_2 + 4x_3 - 8x_4 &= 2 \end{aligned}$$

$$\begin{bmatrix} -1 & 4 & -2 & 4 & 0 \\ 2 & -8 & 1 & -2 & -3 \\ 1 & -4 & 4 & -8 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - 4x_2 = -2$$
$$x_3 - 2x_4 = 1$$
$$x_2 = s \quad x_4 = t$$
$$x_1 = -2 + 4s$$
$$x_3 = 1 + 2t$$

**Question 25**

Solve the given system of equations using either Gaussian or Gauss-Jordan elimination.

$$\begin{aligned} \frac{1}{2}x_1 + x_2 - x_3 - 6x_4 &= 2 \\ \frac{1}{6}x_1 + \frac{1}{2}x_2 - 3x_4 + x_5 &= -1 \\ \frac{1}{3}x_1 - 2x_3 - 4x_5 &= 8 \end{aligned}$$

$$\begin{bmatrix} \frac{1}{2} & 1 & -1 & -6 & 0 & 2 \\ \frac{1}{6} & \frac{1}{2} & 0 & -3 & 1 & -1 \\ \frac{1}{3} & 0 & -2 & 0 & -4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 & 0 & -12 & 24 \\ 0 & 1 & 2 & -6 & 6 & -10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 - 6x_3 - 12x_5 &= 24 \\ x_2 + 2x_3 - 6x_4 + 6x_5 &= -10 \end{aligned}$$

$$x_3 = r \quad x_4 = s \quad x_5 = t$$

$$x_1 = 24 + 6r + 12t$$

$$x_2 = -10 - 2r + 6s - 6t$$

**Question 26**

Solve the given system of equations using either Gaussian or Gauss-Jordan elimination.

$$\begin{aligned} w + x + 2y + z &= 1 \\ w - x - y + z &= 0 \\ x + y &= -1 \\ w + x + z &= 4 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 1 & 4 \end{bmatrix}$$

No solution.

**Question 27**

Use elementary row operations to reduce the given matrix to row echelon form and reduced row echelon form.

$$\begin{bmatrix} -2 & 6 & -7 \\ 3 & -9 & 10 \\ -6 & 18 & -19 \end{bmatrix} \to \begin{bmatrix} -2 & 6 & -7 \\ 3 & -9 & 10 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} -1 & 3 & -\frac{7}{2} \\ 3 & -9 & 10 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} -1 & 3 & -\frac{7}{2} \\ 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -3 & \frac{7}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} (REF)$$

$$\to \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} (RREF)$$

**Question 28**

Determine if the vector $\vec{v}$ is a linear combination of the remaining vectors.

$$\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix} \quad \vec{u_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{u_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} x\vec{u_1} + y\vec{u_2} + z\vec{u_3} &= \vec{v} \\ x + z &= 4 \\ x + y &= 3 \\ x + y + z &= 4 \\ \begin{bmatrix} 1 & 0 & 1 & 4 \\ 1 & 1 & 0 & 3 \\ 1 & 1 & 1 & 4 \end{bmatrix} &\to \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \\ \therefore 3\vec{u_1} + \vec{u_3} &= \vec{v} \end{aligned}$$

**Question 29**

Determine if the vector $\vec{b}$ is in the span of the columns of the matrix $A$.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix}$$

$$x\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} + z\begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix}$$

$$x + 2y + 3z = 4$$
$$4x + 5y + 6z = 7$$
$$6x + 7y + 8z = 9$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This system has a solution so $\vec{b}$ can be represented as a linear combination of the columns of $A$, therefore it is in the span.

**Question 30**

Describe the span of the given vectors geometrically and algebraically.

$$\vec{u} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$$

The set of all linear combinations of $\vec{u}$ and $\vec{v}$ is a plane containing both $\vec{u}$ and $\vec{v}$ as direction vectors and contains all points described by $s\vec{u} + t\vec{v}$ where $s, t$ are arbitrary constants. The vector equation of this plane is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = s\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$$

**Question 31**

Let $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ be (column) vectors in $\mathbb{R}^n$ and let $A$ be the $n \times m$ matrix $[\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}]$ with these vectors as its columns. Then $\vec{v_1}, \vec{v_2}, \ldots, \vec{v_m}$ are linearly dependent if and only if the homogeneous linear system with augmented matrix $[A][0]$ has a nontrivial solution. Use the theorem above to determine if the set of vectors is linearly dependent.

$$\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 3 & 1 & 0 \\ 2 & 1 & -5 & 0 \\ 1 & 2 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The set of vectors is linearly dependent.

## Question 32

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A$ is invertible if $ad - bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then $A$ is not invertible. Find the inverse of the given matrix (if it exists) using the theorem above.

$$\begin{bmatrix} 4 & 19 \\ 1 & 5 \end{bmatrix}$$

$$ad - bc = 20 - 19 = 1$$

$$\begin{aligned}
\begin{bmatrix} 4 & 19 & 1 & 0 \\ 1 & 5 & 0 & 1 \end{bmatrix} &\to \begin{bmatrix} 0 & -1 & 1 & -4 \\ 1 & 5 & 0 & 1 \end{bmatrix} \\
&\to \begin{bmatrix} 0 & -1 & 1 & -4 \\ 1 & 0 & 5 & -19 \end{bmatrix} \\
&\to \begin{bmatrix} 1 & 0 & 5 & -19 \\ 0 & 1 & -1 & 4 \end{bmatrix}
\end{aligned}$$

## Question 33

Use the Gauss-Jordan method to find the inverse of the given matrix (if it exists).

$$\begin{bmatrix} 3 & 2 & 0 \\ -2 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$$

$$\begin{aligned}
\begin{bmatrix} 3 & 2 & 0 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 & 1 & 0 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 3 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 1 \\ 2 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 6 & 3 & -3 & 0 & 0 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 3 & 2 & 0 & -3 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 4 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 6 & 0 & 0 & -6 & -12 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 2 & 3 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & -1 & -2 & 0 \\ 0 & 1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}
\end{aligned}$$

**Question 34**

Determine whether $\vec{b}$ is in $col(A)$. Determine whether $\vec{w}$ is in $row(A)$.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \vec{w} = \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 4 \\ 1 & 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

This augmented matrix has a solution, so $\vec{b}$ is a linear combination of the columns of $A$, therefore it is in the column space of $A$.

$$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

This system does not have a solution, so $\vec{w}$ is not a linear combination of the rows of $A$, therefore it is not in the row space of $A$.

**Question 35**

Determine whether $\vec{b}$ is in $col(A)$. Determine whether $\vec{w}$ is in $row(A)$.

$$A = \begin{bmatrix} 1 & 1 & -5 \\ 0 & 2 & 1 \\ 1 & -1 & 6 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 2 & 4 & -9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -5 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & -1 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This augmented matrix has a solution, so $\vec{b}$ is a linear combination of the columns of $A$, therefore it is in the column space of $A$.

$$\begin{bmatrix} -5 & 1 & 6 & -9 \\ 1 & 2 & -1 & 4 \\ 1 & 0 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

This augmented matrix has a solution, so $\vec{w}$ is a linear combination of the rows of $A$, therefore it is in the row space of $A$.

**Question 36**

If $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$, is $\vec{v} = \begin{bmatrix} -1 \\ 4 \\ -1 \end{bmatrix}$ in $null(A)$?

$$\begin{aligned} A\vec{v} &= \begin{bmatrix} (1)(-1) + (0)(4) + (-1)(-1) \\ (1)(-1) + (1)(4) + (1)(-1) \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{aligned}$$

$\vec{v}$ is not in the null space of $A$.

**Question 37**

Give bases for $row(A), col(A), null(A)$.

$$A = \begin{bmatrix} 2 & -4 & 2 & 1 & 0 \\ -1 & 2 & 1 & 3 & 2 \\ 1 & -2 & 3 & 4 & 2 \end{bmatrix}$$

$$RREF(A) = \begin{bmatrix} 1 & -2 & 0 & -\frac{5}{4} & -1 \\ 0 & 0 & 1 & \frac{7}{4} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$col(A) = span\left(\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}\right)$$

$$x_1 - 2x_2 - \frac{5}{4}x_4 - x_5 = 0$$

$$x_3 + \frac{7}{4}x_4 + x_5 = 0$$

$$x_2 = r \quad x_4 = s \quad x_5 = t$$

$$x_1 = 2r + \frac{5}{4}s + t$$

$$x_3 = -\frac{7}{4}s - t$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$$null(A) = span\left(\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{5}{4} \\ 0 \\ -\frac{7}{4} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

**Question 38**

Find a basis $\mathbb{B}$ for the span of the given vectors.

$$\begin{bmatrix} 4 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$RREF\left(\begin{bmatrix} 4 & -5 & 1 \\ 1 & -1 & 0 \\ 2 & -2 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbb{B} = span\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

## Question 39

Give the rank and nullity of the matrix.

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix}$$

$$RREF(A) = \begin{bmatrix} 1 & 0 & -\frac{7}{2} \\ 0 & 2 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$rank(A) = 2$$

$$nullity(A) = 1$$

## Question 40

Give the rank and nullity of the matrix below.

$$A = \begin{bmatrix} 2 & -5 & 0 & 2 & 1 \\ -1 & 2 & 1 & 2 & 4 \\ 1 & -3 & 1 & 4 & 5 \end{bmatrix}$$

$$RREF(A) = \begin{bmatrix} 1 & 0 & -5 & -14 & -22 \\ 0 & 1 & -2 & -6 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$rank(A) = 2$$

$$x_1 - 5x_3 - 14x_4 - 22x_5 = 0$$

$$x_2 - 2x_3 - 6x_4 - 9x_5 = 0$$

$$nullity(A) = 3 \quad (free\ variables: x_3, x_4, x_5)$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech