Advanced Linear Algebra

Alvin Lin

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Singular Value Decomposition

Recall that the diagonalizaton of A can be written as:

$$A = \begin{bmatrix} \vec{v_1} & \dots & \vec{v_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & \vdots \\ \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \vec{v_1} & \dots & \vec{v_n} \end{bmatrix}^{-1}$$

If A is symmetric, we can orthogonally diagonalize it as $A = PDP^T$.

$$A = \begin{bmatrix} \hat{v_1} & \dots & \hat{v_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & \vdots \\ \dots & \lambda_n \end{bmatrix} \begin{bmatrix} \dots & \vec{v_1}^T & \dots \\ \dots & \vec{v_n}^T & \dots \end{bmatrix}^{-1}$$

Recall from the definition of singular values of A:

$$(A^T A)\hat{v_i} = \lambda_i \hat{v_i} \quad (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$$

We define the singular values $\sigma_i = \sqrt{\lambda_i} = \|A\hat{v_i}\|$.

$$\begin{aligned} \|A\hat{v_i}\|^2 &= (A\hat{v_i})^T (A\hat{v_i}) \\ &= \hat{v_i}^T A^T A \hat{v_i} \\ &= \lambda(\hat{v_i}^T \hat{v_i}) = \lambda \end{aligned}$$

Singular Value Decomposition

$$\begin{aligned} A &= U\Sigma V^T \\ [A] &= \begin{bmatrix} \hat{u_1} & \dots & \hat{u_n} \end{bmatrix} \begin{bmatrix} \sigma_1 & \vdots \\ \dots & \sigma_n \end{bmatrix} \begin{bmatrix} \dots & \hat{v_1}^T & \dots \\ \dots & \hat{v_n}^T & \dots \end{bmatrix} \end{aligned}$$

1. Find the eigenvalues λ_i and eigenvectors $\hat{v}_i$ of $A^T A$.
2. Let $\sigma_i = \sqrt{\lambda_i}$ and normalize $\hat{v}_i$.
3. Let $\hat{u}_i = \frac{1}{\sigma_i} A\hat{v}_i$

Why does this work?

$$\begin{aligned}
AV &= U\Sigma \\
A\begin{bmatrix}\hat{v}_1 & \dots & \hat{v}_n\end{bmatrix} &\stackrel{?}{=} \begin{bmatrix}\hat{u}_1 & \dots & \hat{u}_n\end{bmatrix}\Sigma \\
\begin{bmatrix}A\hat{v}_1 & \dots & A\hat{v}_n\end{bmatrix} &\stackrel{?}{=} \begin{bmatrix}\sigma_1\hat{u}_1 & \dots & \sigma_n\hat{u}_n\end{bmatrix} \\
\begin{bmatrix}\sigma_1\hat{u}_1 & \dots & \sigma_n\hat{u}_n\end{bmatrix} &= \begin{bmatrix}\sigma_1\hat{u}_1 & \dots & \sigma_n\hat{u}_n\end{bmatrix}
\end{aligned}$$

Observations:

- V is orthogonal because the columns are eigenvectors of the symmetric matrix $A^T A$.
- U is orthogonal because the columns $\hat{u}_i$ and $\hat{u}_j$ are orthogonal.

$$\begin{aligned}
\hat{u}_i \cdot \hat{u}_j &= \left(\frac{1}{\sigma_i} A\hat{v}_i\right) \cdot \left(\frac{1}{\sigma_j} A\hat{v}_j\right) \\
&= \frac{1}{\sigma_i\sigma_j}(A\hat{v}_j)^T (A\hat{v}_i) \\
&= \frac{1}{\sigma_i\sigma_j}\hat{v}_j{}^T (A^T A\hat{v}_i) \\
&= \frac{\lambda_i}{\sigma_i\sigma_j}\hat{v}_j{}^T \hat{v}_i \\
&= 0
\end{aligned}$$

In general, $A = U\Sigma V^T$ where

- A is an $m \times n$ matrix
- U is an $m \times m$ orthogonal matrix
- Σ is an $m \times n$ matrix
- V^T is an $n \times n$ orthogonal matrix

Example

Find the singular value decomposition of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$A = U\Sigma V^T$$

$$A^T A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda_1 = 2 \quad \sigma_1 = \sqrt{2} \quad \hat{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$\lambda_2 = 1 \quad \sigma_2 = \sqrt{1} \quad \hat{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_3 = 0 \quad \sigma_3 = 0 \quad \hat{v}_3 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\hat{u}_1 = \frac{1}{\sigma_1} A\hat{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\hat{u}_2 = \frac{1}{\sigma_2} A\hat{v}_2 = \frac{1}{1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech