Advanced Linear Algebra

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Singular Values of a Matrix

Suppose we have an $m \times n$ matrix A, and we construct an $n \times n$ symmetric matrix A^TA. Let λ be an eigenvalue of A^TA with unit eigenvector $\hat{v}$.

$$0 \leq \|A\hat{v}\|^2 = (A\hat{v}) \cdot (A\hat{v}) = (A\hat{v})^T(A\hat{v}) = \hat{v}^TA^TA\hat{v} = \hat{v}^T\lambda\hat{v} = \lambda\|\hat{v}\|^2 = \lambda$$

If A is an $m \times n$ matrix, the "singular values" of A are the square roots of the eigenvalues of A^TA and are denoted $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$. Let $\{\hat{v_1}, \ldots, \hat{v_n}\}$ be orthonormal eigenvectors of A^TA such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. We know $\lambda_i = \|A\hat{v_i}\|^2$ and $\sigma_i = \|A\hat{v_i}\| = \sqrt{\lambda_i}$.

Example

Suppose we have the matrix:

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The eigenvalues of $A^T A$ are $\lambda_1 = 3$ and $\lambda_2 = 1$ and thus the singular values of $A^T A$ are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = \sqrt{1} = 1$. The eigenvectors of $A^T A$ are:

$$\lambda_1 = 3 \quad \hat{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\lambda_2 = 1 \quad \hat{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Consider a vector in $\mathbb{R}^2$

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

such that $\|\hat{x}\| = 1$.

$$\begin{aligned} \|A\hat{x}\|^2 &= (A\hat{x}) \cdot (A\hat{x}) \\ &= (A\hat{x})^T (A\hat{x}) \\ &= \hat{x}^T A^T A \hat{x} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= 2x_1^2 + 2x_1 x_2 + 2x_2^2 \end{aligned}$$

We are interested in what the largest or smallest that the quadratic polynomial can become, which represents the largest or smallest amount that the matrix A can expand or shrink the vector.

Facts

If $\|\vec{x}\| = 1$ and $f(\vec{x}) = \vec{x}^T B \vec{x}$ where B is a symmetric matrix:

1) The maximum $f(\vec{x})$ is λ_1 and occurs at $\vec{x} = \hat{v}_1$.

2) The minimum $f(\vec{x})$ is λ_n and occurs at $\vec{x} = \hat{v}_n$.

Continuing the previous example, the maximum of $\|A\hat{x}\|^2 = \lambda_1 = 3$. The max $\|A\hat{x}\| = \sqrt{3} = \sigma_1$, which occurs at $\hat{v}_1$, and the min $\|A\hat{x}\| = \sqrt{1} = \sigma_2$, which occurs at $\hat{v}_2$. If we consider A as a mapping from one space to another in $\mathbb{R}^3$:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_1 \\ x_2 \end{bmatrix}$$

A maps all of $\mathbb{R}^2$ to the plane $x-y-z=0$. In particular, the vectors on the unit circle get mapped to an ellipse on this plane. The maximum amount of stretching that A can do with respect to the previous example is $\sqrt{3}$, occurring at the vector $\hat{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$ on the unit circle. The minimum amount of stretching occurs at $\hat{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech