Advanced Linear Algebra

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Inner Product Spaces

An inner product is an operation that assigns to every pair u, v in V a real number $\langle u, v\rangle$ such that:

1. $\langle u, v\rangle = \langle v, u\rangle$
2. $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$
3. $\langle cu, v\rangle = c\langle u, v\rangle$
4. $\langle u, v\rangle \geq 0$ and $\langle u, u\rangle = 0$ if $u = 0$

$\langle \vec{u}, \vec{v}\rangle = \vec{u} \cdot \vec{v}$ is an inner product. Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. We can show that $\langle u, v\rangle = 2u_1v_1 + 3u_2v_2$ defines an inner product.

1. This satisfies the first property by inspection.
2. By the following, this definition satisfies the second property:

$$\begin{aligned} \langle u, v + w\rangle &= 2u_1(v_1 + w_1) + 3u_2(v_2 + w_3) \\ &= 2u_1v_1 + 3u_2v_2 + 2u_1w_1 + 3u_2w_2 \\ &= \langle u, v\rangle + \langle u, w\rangle \end{aligned}$$

3. This satisfies the third property by inspection.
4. This satisfies the fourth property by inspection.

Weighted Dot Product

The weighted dot product $\langle u, v\rangle$ is defined as:

$$\begin{aligned}\langle \vec{u}, \vec{v}\rangle &= w_1u_1v_1 + \cdots + w_nu_nv_n \\ &= \vec{u}^T W\vec{v}\end{aligned}$$

where W is a matrix with the weights along the diagonal and everything else is 0.

Example

Let f, g in $C[a, b]$.

$$\langle f, g\rangle = \int_a^b f(x)g(x)\ \mathrm{d}x$$

1. This satisfies the first property by inspection.
2. $\langle f, g+h\rangle = \int_a^b f(g+h)\ \mathrm{d}x = \int_a^b fg\ \mathrm{d}x + \int_a^b fh\ \mathrm{d}x = \langle f, g\rangle + \langle f, h\rangle$
3. This satisfies the third property by inspection.
4. This satisfies the fourth property by inspection.

Definitions

1. The length (or norm) is $\|\vec{v}\| = \sqrt{\langle v, v\rangle}$
2. The distance is $d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$
3. $\vec{u}$ and $\vec{v}$ are orthogonal if $\langle u, v\rangle = 0$

Suppose in $C[0, 1]$, we define $f(x) = x, g(x) = 3x - 2$.

1. $\|f\|^2 = \langle f, f\rangle = \int_0^1 f^2\ \mathrm{d}x = \int_0^1 x^2\ \mathrm{d}x = \frac{1}{3}$
2. $d(f, g) = \|f - g\| = \sqrt{\langle f-g, f-g\rangle} = \sqrt{\int_0^1 (f-g)^2\ \mathrm{d}x} = \sqrt{\int_0^1 (-2x+2)^2\ \mathrm{d}x} = \frac{2}{\sqrt{3}}$
3. $\langle f, g\rangle = \int_0^1 fg\ \mathrm{d}x = \int_0^1 x(3x-2)\ \mathrm{d}x = \int_0^1 (3x^2 - 2x)\ \mathrm{d}x = [x^2 - x^2]_0^1 = 0$
   (f and g are orthogonal)

Pythagoras' Theorem

$$\begin{aligned}\|u+v\|^2 &= \langle u+v, u+v\rangle \\ &= \langle u,u\rangle + 2\langle u,v\rangle + \langle v,v\rangle \\ &= \|u\|^2 + 2\langle u,v\rangle + \|v\|^2\end{aligned}$$

Example

Construct an orthogonal basis for P_2 with respect to the inner product

$$\langle f,g\rangle = \int_{-1}^{1} f(x)g(x)\ \mathrm{d}x$$

by applying Gram-Schmidt to $\{1, x, x^2\}$.

$$\begin{aligned}
v_1 &= x_1 = 1 \quad W_1 = span(v_1) \\
v_2 &= x_2 - \mathrm{proj}_{W_1} x_2 \\
&= x_2 - \left(\frac{\langle x_1, x_2\rangle}{\langle v_1, v_1\rangle}\right) \\
&= x_2 - \left(\frac{\int_{-1}^{1}(1)(x)\ \mathrm{d}x}{\int_{-1}^{1}(1)^2\ \mathrm{d}x}\right) \\
&= x_2 - \left(\frac{0}{2}\right)(1) \\
&= x \quad W_2 = span(1, x) \\
v_3 &= x_3 - \mathrm{proj}_{W_2} x_3 \\
&= x_3 - \frac{\langle v_1, x_3\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle v_2, x_3\rangle}{\langle v_2, v_3\rangle} v_2 \\
&= x^2 - \left(\frac{\frac{2}{3}}{2}\right) 1 - \left(\frac{0}{\frac{2}{3}}\right) x \\
&= x^2 - \frac{1}{3} \\
P_2 &= span\left\{1, x, x^2 - \frac{1}{3}\right\}
\end{aligned}$$

Historical note: these three elements for the basis of P_2 are the first three Legendre polynomials.

Cauchy-Schwarz Inequality and Triangle Inequality

$$|\langle u, v\rangle| \leq \|u\|\|v\|$$

In abstract vector spaces, the dot product $\mathbb{R}^n$ is

$$\vec{u} \cdot \vec{v} = \|u\|\|v\| \cos\theta$$

The triangle inequality generalizes to

$$\|u + v\| \leq \|u\| + \|v\|$$

In summary, inner products allow us to generalize many properties we used in $\mathbb{R}^2$ to abstract vector spaces.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech