Advanced Linear Algebra

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Linear Transformations

A linear transformation $T : V \to W$ is a mapping such that

1. $T(u + v) = T(u) + T(v)$
2. $T(vu) = cT(u)$

In a more compressed form

$$T(c_1u_1 + \cdots + c_ku_k) = c_1T(u_1) + \cdots + c_kT(u_k)$$

For example, consider the transformation $T : M_{mn} \to M_{mn}$ where $T(A) = A^T$.

1. $T(A + B) = (A + B)^T = A^T + B^T = T(A) + T(B)$
2. $T(cA) = (cA)^T = cA^T = cT(A)$

This also applies for transformations in the space of functions. Suppose we have $D : D \to F, D(f) = f'$.

1. $D(f + g) = (f + g)' = f' + g' = D(f) + D(g)$
2. $D(cf) = (cf)' = cf' = cD(f)$

As another example, consider the transformation $S : C[a, b] \to \mathbb{R}$ where $S(f) = \int_a^b f \ \mathrm{d}x$.

1. $S(f + g) = \int_a^b (f + g) \ \mathrm{d}x = \int_a^b f \ \mathrm{d}x + \int_a^b g \ \mathrm{d}x$
2. $S(cf) = \int_a^b cf \ \mathrm{d}x = cS(f)$

Now consider the transformation $T : M_{22} \to \mathbb{R}, T(A) = \det(A)$.

$$\begin{aligned} T(A+B) &= \det(A+B) \\ &\neq \det(A) + \det(B) \\ &\neq T(A) + T(B) \end{aligned}$$

Thus, this transformation is not a linear transformation.

Zero and Identity Transformation

The zero and identity transformations are special transformations:

- Zero transformation: $T_0(A) = 0$
- Identity transformation: $T_I(A) = A$

Example

Given the linear transformation $T : \mathbb{R}^2 \to P_2$ and

$$\begin{aligned} T\begin{bmatrix}1\\1\end{bmatrix} &= 2 - 3x + x^2 \\ T\begin{bmatrix}2\\3\end{bmatrix} &= 1 - x^2 \end{aligned}$$

Find $T\begin{bmatrix}-1\\2\end{bmatrix}$.

Note that $B = \left\{\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}2\\3\end{bmatrix}\right\}$ is a basis for $\mathbb{R}^2$.

$$\begin{aligned} \begin{bmatrix}-1\\2\end{bmatrix} &= -7\begin{bmatrix}1\\1\end{bmatrix} + 3\begin{bmatrix}2\\3\end{bmatrix} \\ T\begin{bmatrix}-1\\2\end{bmatrix} &= -7T\begin{bmatrix}1\\1\end{bmatrix} + 3T\begin{bmatrix}2\\3\end{bmatrix} \\ &= -7(2 - 3x + x^2) + 3(1 - x^2) \\ &= -11 + 21x - 10x^2 \end{aligned}$$

Composition

Suppose we have the transformations $T : U \to V$ and $S : V \to W$. The composition of the transformations $(S \circ T)u = S(T(u))$. For example:

$$\begin{aligned} T : \mathbb{R}^2 \to P_1, T \begin{bmatrix} a \\ b \end{bmatrix} &= a + (a+b)x \\ S : P_1 \to P_2, S(p) &= xp \\ (S \circ P) \begin{bmatrix} 3 \\ -2 \end{bmatrix} &= S\left(T \begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) \\ &= S(3 + x) \\ &= x(3 + x) \\ &= 3x + x^2 \end{aligned}$$

Inverse Transformations

The definition of an inverse transformation T' is that $T' \circ T = Iv$. For example:

$$\begin{aligned} T : \mathbb{R}^2 \to P_1, T \begin{bmatrix} a \\ b \end{bmatrix} &= a + (a+b)x \\ T' : P_1 \to \mathbb{R}_2, T'(c + dx) &= \begin{bmatrix} c \\ d - c \end{bmatrix} \\ (T' \circ T) \begin{bmatrix} a \\ b \end{bmatrix} &\stackrel{?}{=} \begin{bmatrix} a \\ b \end{bmatrix} \\ T'\left(T \begin{bmatrix} a \\ b \end{bmatrix}\right) &= T'(a + (a+b)x) \\ &= \begin{bmatrix} a \\ a + b - a \end{bmatrix} \\ &= \begin{bmatrix} a \\ b \end{bmatrix} \end{aligned}$$

Kernel and Range of $T : V \to W$

The kernel and range of a transformation T are defined as follows:

$$\begin{aligned} ker(T) &= \{v \in V : T(v) = 0\} \\ range(T) &= \{T(v) : v \in V\} \\ &= \{w \in W : w = T(v) \text{ for some } v \in V\} \end{aligned}$$

For example, given the transformation $T : \mathbb{R}^n \to \mathbb{R}^m, T(v) = Av$:

$$range(T) = \{Av : v \in \mathbb{R}^n\} = col(A)$$
$$ker(T) = \{v \in \mathbb{R}^n : Av = 0\} = null(A)$$

The kernel and the range are generalizations of the null space and the column space.

$$rank(T) = dim(range(T))$$
$$nullity(T) = dim(ker(T))$$
$$rank(T) + nullity(T) = dim(V)$$

Example

Suppose have the transformation $D : P_3 \to P_2, Dp = p'$. Find the kernel of the linear operator D.

$$ker(D) = \{p \in P_3 : Dp = 0\}$$
$$= \{a : a \in \mathbb{R}\}$$

Since the derivative of any constant is 0, the kernel of the linear operator is the set of real numbers.
Find the range of D.

$$range(D) = \{Dp : p \in P_3\} = P_2$$

Find the rank and nullity of D.

$$rank(D) = dim(P_2) = 3$$
$$nullity(D) = dim(ker(D)) = 1$$

Example

Given the transformation $S : P_1 \to \mathbb{R}, S(p(x)) = \int_0^1 p(x)\ \mathrm{d}x$, find $ker(S)$.

$$
\begin{aligned}
Let : p &= a + bx \text{ in } P_1 \\
Sp &= \int_0^1 (a + bx)\ \mathrm{d}x \\
&= \left[ax + \frac{b}{2}x^2\right]_0^1 \\
&= a + \frac{b}{2} \\
ker(s) &= \{p \in P_1 : Sp = 0\} \\
&= \left\{a + bx \in P_1 : a + \frac{b}{2} = 0\right\} \\
&= \left\{-\frac{b}{2} + bx\right\}
\end{aligned}
$$

Find the rank and nullity of S.

$$
\begin{aligned}
basis(ker(S)) &= span\left(-\frac{1}{2} + x\right) \\
nullity(S) &= dim(ker(S)) = 1 \\
rank(S) &= dim(range(S)) = dim(\mathbb{R}) = 1
\end{aligned}
$$

One to One Mappings

$T : V \to W$ is called "one-to-one" if it maps distinct vectors in V to distinct vectors in W. If $range(T) = W$, then T is "onto".
Theorem: A linear transformation $T : V \to W$ is 1-1 if and only if $ker(T) = \{0\}$. To prove this, assume T is 1-1 and that $v \in ker(T)$.

$$
\begin{aligned}
T(v) &= 0 \quad T(0) = 0 \\
\therefore T(v) &= T(0)
\end{aligned}
$$

Since T is 1-1, then $v = 0$. Conversely, assume $ker(T) = \{0\}$. Let u, v in V with $T(u) = T(v)$.

$$T(u - v) = T(u) - T(v) = 0$$

If this statement is true, then $u - v$ is in $ker(T)$, and since $ker(T) = \{0\}$:

$$u - v = 0 \quad u = v$$

Therefore, T is 1-1.
Theorem: $T : V \to W$. Let $dim(V) = dim(W) = n$, then T is 1-1 if and only if it is onto.
Theorem: Let $T : V \to W$ be 1-1. If $S = \{v_1, \ldots, v_n\}$ is linearly independent in V then $T(s) = \{T(v_1), \ldots, T(v_n)\}$ is linearly independent in W.
Theorem: $T : V \to W$ is invertible if and only if it is 1-1 and onto.

Example

Given the transformation $T : \mathbb{R}^2 \to \mathbb{R}^3, T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ x - y \\ 0 \end{bmatrix}$, is T 1-1? Does $T(u) = T(v)$ imply $u = v$?

$$T \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = T \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \stackrel{?}{\Rightarrow} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ x_1 - y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 - y_2 \\ 0 \end{bmatrix}$$

$$2x_1 = 2x_2 \to x_1 = x_2$$

$$x_1 - y_1 = x_2 - y_2 \to y_1 = y_2$$

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

T is one to one, but not onto because

$$T \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ for any } \begin{bmatrix} x \\ y \end{bmatrix}$$

Example

$T : \mathbb{R}^2 \to P_1, T \begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x$. Show that T is 1-1 and onto.

Let $\begin{bmatrix} a \\ b \end{bmatrix}$ be in $ker(T)$.

$$T \begin{bmatrix} a \\ b \end{bmatrix} = a + (a+b)x = 0$$
$$a = 0 \quad a + b = 0$$
$$b = -a \quad b = 0$$
$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = ker(T)$$

Since $ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$, it is 1-1. We can prove T is onto using the rank theorem.

$$rank(T) + nullity(T) = dim(V)$$
$$rank(T) + 0 = 2$$
$$dim(range(T)) = 2$$
$$rank(T) = dim(range(T))$$

Therefore, T is onto.

Isomorphism

$T : V \to W$ is called an "isomorphism" if it is 1-1 and onto. If V and W are vector spaces such that there is an isomorphism from V to W, then we say V and W are "isomorphic" ($V \cong W$).

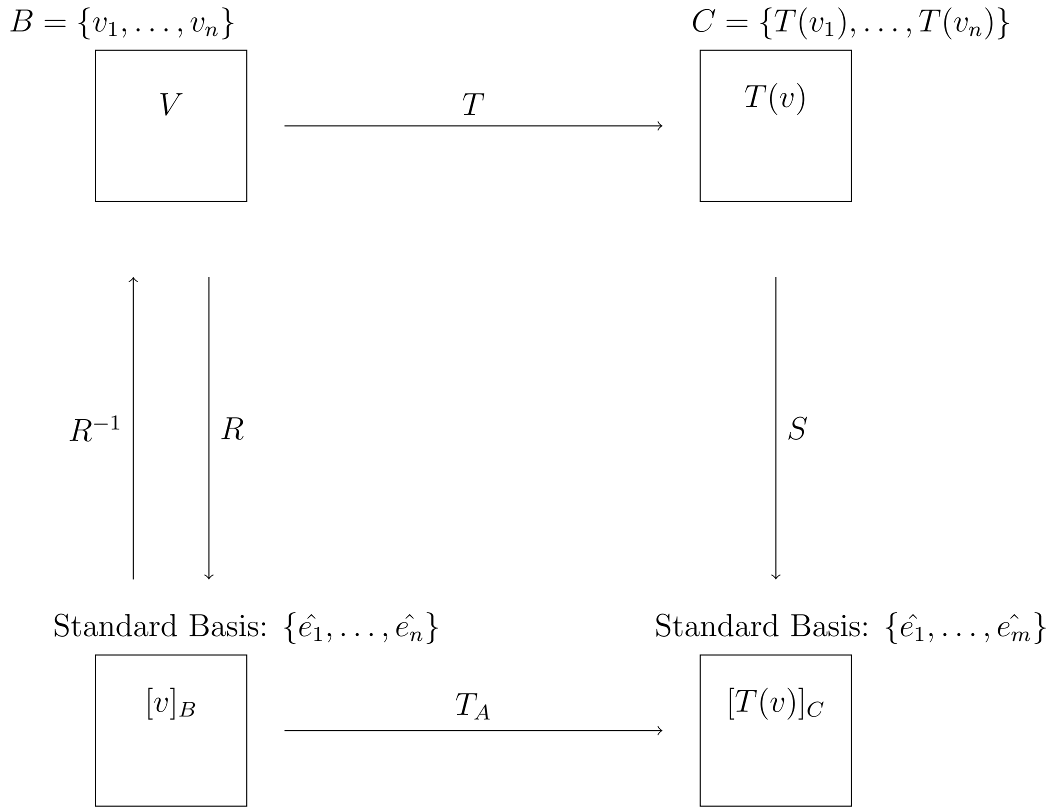
Example

Suppose P_{n-1} and $\mathbb{R}^n$ are isomorphic.

$$T(a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}) = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

Note that we know T is 1-1 since $ker(T) = \{0\}$. Therefore, $P_{n-1} \cong \mathbb{R}^n$.

Matrix of Linear Transformation



$$
\begin{aligned}
[T(v)]_C &= A[v]_B \\
&= (S \circ T \circ R^{-1})[v]_B \\
A &= \Big[[T(v_1)]_C, T(v_2)_C, \ldots, [T(v_n)]_C \Big] \\
&= [T]_{C \leftarrow B}
\end{aligned}
$$

Every linear transformation can be represented as a matrix multiplication under some given bases.

Example

Suppose we have the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2, T \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y \\ x + y - 3z \end{bmatrix}$. Let $B = \{\hat{e_1}, \hat{e_2}, \hat{e_3}\}$ and $C = \{\hat{e_2}, \hat{e_1}\}$.

$$T(\hat{e_1}) = T \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad [T(\hat{e_1})]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_C$$

$$T(\hat{e_2}) = T \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad [T(\hat{e_2})]_C = \begin{bmatrix} 1 \\ -2 \end{bmatrix}_C$$

$$T(\hat{e_3}) = T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix} \quad [T(\hat{e_3})]_C = \begin{bmatrix} -3 \\ 0 \end{bmatrix}_C$$

$$A = \begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

We can verify that this works for $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$.

$$T \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} -5 \\ 10 \end{bmatrix}$$

$$[v]_B = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$$

$$A[v]_B = [T(v)]_C$$

$$\begin{bmatrix} 1 & 1 & -3 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} 10 \\ -5 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

Example

Suppose we have the transformation $D : P_3 \to P_2, D(p(x)) = p'(x)$. Let $B = \{1, x, x^2, x^3\}$ and $C = \{1, x, x^2\}$.

$$D(1) = 0 \quad [D(1)]_C = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$D(x) = 1 \quad [D(x)]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$D(x^2) = 2x \quad [D(x)]_C = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$D(x^3) = 3x^2 \quad [D(x^3)]_C = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Compute $D(5 - x + 2x^3)$.

$$\begin{aligned} D(5 - x + 2x^3) &= -1 + 6x^2 \\ &= \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}_C \end{aligned}$$

We can check that we get the same answer using the matrix $A[v]_B = [D(v)]_C$.

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 6 \end{bmatrix}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech