Advanced Linear Algebra

Alvin Lin

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Changes of Basis

Suppose we have two different bases:

$$\vec{u_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \vec{u_2} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$
$$B = \{\vec{u_1}, \vec{u_2}\}$$
$$\vec{v_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$C = \{\vec{v_1}, \vec{v_2}\}$$

Suppose we have a vector $\vec{x}$ represented in one basis:

$$[\vec{x}]_B = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1\vec{u_1} + 3\vec{u_2} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

The same vector represented with respect to the basis C is:

$$[\vec{x}]_C = \begin{bmatrix} 6 \\ -1 \end{bmatrix} = 6\vec{v_1} - \vec{v_2} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

In the standard basis, this vector is $5\hat{\imath} - \hat{\jmath}$. In general, given a vector $\vec{x}$ and a set of bases, how do we convert the vector between the different bases? We just need to know how to represent the basis vectors in terms of each other in order to convert between them.

$$
\begin{aligned}
\vec{u_1} &= -3\vec{v_1} + 2\vec{v_2} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}_C \\
\vec{u_2} &= 3\vec{v_1} - \vec{v_2} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}_C \\
\vec{x} &= 1\vec{u_1} + 3\vec{u_2} \\
&= 1(-3\vec{v_1} + 2\vec{v_2}) + 3(3\vec{v_1} - \vec{v_2}) \\
&= 6\vec{v_1} - \vec{v_2}
\end{aligned}
$$

We can compactly represent this as:

$$
\begin{aligned}
\begin{bmatrix} 6 \\ -1 \end{bmatrix}_C &= \begin{bmatrix} -3 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}_B \\
\vec{x}_C &= \begin{bmatrix} \vec{u_1}_C & \vec{u_2}_C \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix}_B \\
[\vec{x}]_C &= P_{C\leftarrow B}[\vec{x}]_B
\end{aligned}
$$

Theorem

Let $B = \{\vec{u_1}, \ldots, \vec{u_n}\}$ and $C = \{\vec{v_1}, \ldots, \vec{v_n}\}$ be the bases for some vector space. Then

$$
\begin{aligned}
P_{C\leftarrow B} &= \begin{bmatrix} [\vec{u_1}]_C & \ldots & [\vec{u_n}]_C \end{bmatrix} \\
[\vec{x}]_C &= P_{C\leftarrow B}[\vec{x}]_B \\
[\vec{x}]_B &= (P_{C\leftarrow B})^{-1}[\vec{x}]_C \\
P_{B\leftarrow C} &= (P_{C\leftarrow B})^{-1}
\end{aligned}
$$

The change of basis matrix $P_{C\leftarrow B}$ is unique and can be used to convert between the bases.

Example

$$
\begin{aligned}
B &= \{1, x, x^2\} \\
C &= \{1 + x, x + x^2, 1 + x^2\} \\
p = 1 + 2x - x^2 &= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}_B
\end{aligned}
$$

Represent p in the basis C.

$$\begin{aligned}
[1+x]_B &= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_B \\
[x+x^2]_B &= \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}_B \\
[1+x^2]_B &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}_B \\
P_{B\leftarrow C} &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \\
[p]_B &= (P_{B\leftarrow C})[p]_C \\
[p]_C &= (P_{B\leftarrow C})^{-1}[p]_B \\
&= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}_B \\
&= \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}_C
\end{aligned}$$

Example

$$\begin{aligned}
B &= \{E_{11}, E_{21}, E_{12}, E_{22}\} \\
&= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \\
C &= \{A, B, D, E\} \\
&= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}
\end{aligned}$$

Find the change of basis matrix.

$$E_{11} = 1A \quad [E_{11}]_C = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$E_{21} = -B + D \quad [E_{21}]_C = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$E_{12} = -A + B \quad [E_{12}]_C = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$E_{22} = E - D \quad [E_{22}]_C = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

$$P_{C \leftarrow B} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Suppose we have the matrix

$$\vec{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1E_{11} + 3E_{21} + 2E_{12} + 4E_{22} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}_B$$

Find the coordinates of $\vec{x}$ in terms of the basis C.

$$[\vec{x}]_C = P_{C \leftarrow B} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \end{bmatrix}_B$$

$$= \begin{bmatrix} -1 \\ -1 \\ -1 \\ 4 \end{bmatrix}_C$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech