Advanced Linear Algebra

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January 2019 - May 2019

Abstract Vector Spaces

Recall the definition of linear dependence:

$$\begin{aligned} c_1v_1 + \cdots + c_nv_n &= 0 \\ c_1 = c_2 = \ldots &= 0 \\ v_1, \ldots, v_n &\text{ are linearly dependent} \end{aligned}$$

Example

$$\{\sin^2(x), \cos^2(x), \cos(2x)\}$$

Since $\cos(2x) = \cos^2(x) - \sin^2(x)$, this set is linearly dependent.

Example

$$\{1 + x, x + x^2, 1 + x^2\}$$

If the equation

$$c_1(1 + x) + c_2(x + x^2) + c_3(1 + x^2) = 0$$

has a nontrivial solution for c_1, c_2, c_3, then one element can be written as a linear combination of the other elements, and thus the set is linearly dependent.

$$\begin{aligned} (c_1 + c - 3) + (c_1 + c_3)x + (c_2 + c_3)x^2 &= 0 \\ c_1 + c_3 &= 0 \\ c_1 + c_2 &= 0 \\ c_2 + c_3 &= 0 \end{aligned}$$

We must examine this system of linear equations to determine if it has a nontrivial solution. We can so this using an augmented matrix.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right]$$

Since this augmented matrix has only the trivial solution $c_1 = c_2 = c_3 = 0$, the set is linearly independent. Recall that $\mathbb{B}$ is a basis for V if $\mathbb{B}$ spans V and $\mathbb{B}$ is linearly independent. We know that $\mathbb{B}$ spans V if

$$a + bx + cx^2 = c_1v_1 + c_2v_2 + c_3v_3$$

We can use the augmented matrix from before to construct the following matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 0 & b \\ 0 & 1 & 1 & c \end{array}\right]$$

Since the rank of the matrix is 3, the matrix is invertible, which means there exist a unique c_1, c_3, c_3 for any a, b, c. Therefore, the set $\{1 + x, x + x^2, 1 + x^2\}$ is a basis for the polynomial space P_2.

Example

Find a basis for $W_2 = \{a + bx - bx^2 + ax^3\}$ where a and b are arbitrary constants.

$$\begin{aligned} a + bx - bx^2 + ax^3 &= a(1 + x^3) + b(x - x^2) \\ &= au + bv \\ u = 1 + x^3 &\quad v = x - x^2 \\ W_2 &= span\{1 + x^3, x - x^2\} \end{aligned}$$

Example

Find a basis for $W_3 = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

$$\begin{aligned} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} &= a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\ W_2 &= span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\} \end{aligned}$$

Standard Basis

$$\begin{aligned}
\mathbb{R}^n &= span\{\hat{e_1}, \hat{e_2}, \dots, \hat{e_n}\} \\
P^n &= span\{1, x, x^2, \dots, x^n\} \\
M_{mn} &= span\{E_{11}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{m1}, \dots, E_{mn}\}
\end{aligned}$$

Coordinates

Let $\mathbb{B} = \{v_1, \dots, v_n\}$ and let $v = c_1 v_2 + \cdots + c_n v_n$. Then the coordinate representation $[v]_{\mathbb{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$. For example, if we have the space P_2 whose basis is $\mathbb{B} = \{1, x, x^2\}$ and a polynomial $p(x) = 2 - 3x + 5x^2$, then $[p]_{\mathbb{B}} = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$.

Example

Suppose we have the space P_2 with basis $\mathbb{B} = \{1+x, x+x^2, 1+x^2\}$ and $p = 1+2x-x^2$. Find $[p]_{\mathbb{B}}$.

$$\begin{aligned}
c_1(1+x) + c_2(x+x^2) + c_3(1+x^2) &= 1 + 2x - x^2 \\
(c_1 + c_3) + (c_1 + c_2)x + (c_2 + c_3)x^2 &= 1 + 2x - x^2 \\
\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & -1 \end{array}\right] &\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \\
[p]_{\mathbb{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} &= \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}
\end{aligned}$$

Example

Find the dimension of the vector space of symmetric 2×2 matrices.

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$W = span \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\dim(W) = 3$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech