Advanced Linear Algebra

Alvin Lin

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QR Factorization: A = QR

Let A be an $m \times n$ matrix with linearly independent columns.

$$A = \begin{bmatrix} \vec{a_1} & \vec{a_2} & \dots & \vec{a_n} \end{bmatrix}$$

Apply Gram-Schmidt to the column vectors to find an orthonormal basis $\hat{q_1}, \dots, \hat{q_n}$.

$$\begin{aligned} \vec{a_1} &= r_{11}\hat{q_1} \\ \vec{a_2} &= r_{12}\hat{q_1} + r_{22}\hat{q_2} \\ &\vdots \\ \vec{a_n} &= r_{1n}\hat{q_1} + r_{2n}\hat{q_2} + \dots + r_{nn}\hat{q_n} \end{aligned}$$

$$\begin{bmatrix} \vec{a_1} & \vec{a_2} & \dots & \vec{a_n} \end{bmatrix} = \begin{bmatrix} \hat{q_1} & \hat{q_2} & \dots & \hat{q_3} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & \dots & r_{1n} \\ 0 & r_{22} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & r_{nn} \end{bmatrix}$$

Example

Find the QR factorization of

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Apply Gram-Schmidt to the column vectors.

$$
\begin{aligned}
Q &= \begin{bmatrix} \frac{1}{2} & \frac{3}{2\sqrt{5}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{2} & \frac{3}{2\sqrt{5}} & 0 \\ -\frac{1}{2} & \frac{1}{2\sqrt{5}} & \frac{1}{\sqrt{6}} \\ \frac{1}{2} & \frac{1}{2\sqrt{5}} & \frac{2}{\sqrt{6}} \end{bmatrix} \\
A &= QR \\
Q^T A &= Q^T QR \\
R &= Q^T A \\
&= \begin{bmatrix} 2 & 1 & \frac{1}{2} \\ 0 & \sqrt{5} & \frac{3\sqrt{5}}{2} \\ 0 & 0 & \frac{\sqrt{6}}{2} \end{bmatrix}
\end{aligned}
$$

QR Algorithm for Approximating Eigenvalues

$$
\begin{aligned}
\text{Let } A &= QR \\
A_1 &= RQ = Q_1 R_1 \\
A_2 &= R_1 Q_1 = Q_2 R_2 \\
A_3 &= R_2 Q_2 = Q_3 R_3 \\
&\vdots \\
A_k &= R_{k-1} Q_{k-1}
\end{aligned}
$$

Notice that this iterative process yields similar matrices. Therefore, the eigenvalues remain the same. As $k \to \infty$, A_k converges to an upper triangular matrix. Since A_k is similar to A, the eigenvalues of A are the values on the diagonal of A_k.

Orthogonal Diagonalization of Symmetric Matrices

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$
$$D = P^{-1}AP$$
$$D = \begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix}$$
$$\begin{bmatrix} -3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$
$$\vec{v_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \vec{v_1} \perp \vec{v_2}$$
$$\hat{u_1} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{bmatrix} \quad \hat{u_2} = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix}$$
$$Q = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$
$$D = Q^{-1}AQ = Q^TAQ$$

In general, a square matrix is orthogonally orthogonally diagonalizable if $Q^TAQ = D$ where Q is an orthogonal matrix and D is a diagonal matrix.

Theorem

If a matrix A is orthogonally diagonalizable, then A is symmetric. Proof:

$$\begin{aligned} Q^TQA &= D \\ \therefore A &= QDQ^T \\ A^T &= (QDQ^T)^T = (Q^T)^TD^TQ^T = QDQ^T \\ &= A^T \end{aligned}$$

The converse is also true, if a matrix A is symmetric, it is also orthogonally diagonalizable.

Theorem

Additionally, if A is a real symmetric matrix, then the eigenvalues of A are real. Proof:

$$\begin{aligned}
\overline{zw} &= \bar{z}\bar{w} \\
Av &= \lambda v \\
\overline{Av} &= \bar{\lambda} v \\
A\bar{v} &= \bar{A}\bar{v} = \overline{Av} = \overline{\lambda v} = \bar{\lambda}\bar{v} \\
(A\bar{v})^T &= (\bar{\lambda}\bar{v})^T \\
\bar{v}^T A^T &= \bar{\lambda}\bar{v}^T \\
\bar{v}^T A &= \bar{\lambda}\bar{v}^T \\
\lambda\bar{v}^T v &= \bar{v}^T(\lambda v) = \bar{v}^T(Av) = (\bar{v}^T A)v = \bar{\lambda}\bar{v}^T v \\
(\lambda - \bar{\lambda})\bar{v}^T v &= 0 \\
\bar{v}^T v &\neq 0 \\
\lambda &= \bar{\lambda}
\end{aligned}$$

Theorem

If A is a symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues are orthogonal.

$$\begin{aligned}
A\vec{v_1} &= \lambda_1\vec{v_1} \quad A\vec{v_2} = \lambda_2\vec{v_2} \\
\lambda_1(\vec{v_1}\cdot\vec{v_2}) &= (\lambda_1\vec{v_1})\cdot\vec{v_2} = (A\vec{v_1})\cdot\vec{v_1} = (A\vec{v_1})^T\vec{v_2} \\
&= \vec{v_1}^T A^T\vec{v_2} = \vec{v_1}^T(A\vec{v_2}) = \vec{v_1}^T\lambda_2\vec{v_2} = \lambda_2(\vec{v_1}\cdot\vec{v_2}) \\
(\lambda_1 - \lambda_2)(\vec{v_1}\cdot\vec{v_2}) &= 0 \\
(\lambda_1 - \lambda_2) &\neq 0 \\
\vec{v_1}\cdot\vec{v_2} &= 0
\end{aligned}$$

The Spectral Theorem

Let A be an $n \times n$ real matrix. Then A is symmetric if and only if it is orthogonally diagonalizable $Q^T AQ = D$. Example:

Orthogonally diagonalize A:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$\lambda_1 = 4 \quad \lambda_2 = \lambda_3 = 1$$

$$E_4 = span\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} (\vec{x_1})$$

$$E_1 = span\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\} (\vec{x_2}, \vec{x_3})$$

$$\vec{x_1} \cdot \vec{x_2} = 0 \quad \vec{x_1} \cdot \vec{x_3} = 0$$

$$\vec{x_2} \cdot \vec{x_3} = 1 \neq 0$$

$$\text{Let } \vec{v_1} = \vec{x_1}$$

$$\vec{v_2} = \vec{x_2}$$

$$\vec{v_3} = \text{perp}_{\vec{x_2}} \vec{x_3} = x_3 - \text{proj}_{\vec{x_2}} \vec{x_3} = \vec{x_3} - \left(\frac{\vec{x_3} \cdot \vec{x_2}}{\vec{x_2} \cdot \vec{x_2}} \right) = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

$$\|\vec{v_3}\| = \frac{\sqrt{6}}{2}$$

$$\hat{q_1} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \hat{q_2} = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \hat{q_3} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \end{bmatrix}$$

$$Q = \begin{bmatrix} \hat{q_1} & \hat{q_2} & \hat{q_3} \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Spectral Decomposition of $A = QDQ^T$

$$
\begin{aligned}
A &= \begin{bmatrix} \hat{q_1} & \dots & \hat{q_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & \vdots \\ \vdots & \lambda_n \end{bmatrix} \begin{bmatrix} \hat{q_1}^T \\ \vdots \\ \hat{q_n}^T \end{bmatrix} \\
&= \begin{bmatrix} \lambda_1 \hat{q_1} & \dots & \lambda_n \hat{q_n} \end{bmatrix} \begin{bmatrix} \hat{q_1}^T \\ \vdots \\ \hat{q_n}^T \end{bmatrix} \\
&= \lambda_1 \hat{q_1} \hat{q_1}^T + \dot{+} \lambda_n \hat{q_n} \hat{q_n}^T
\end{aligned}
$$

$\lambda_1 \hat{q_1} \hat{q_1}^T$ is a projection onto the subspace spanned by $\hat{q_1}$, and $\lambda_n \hat{q_n} \hat{q_n}^T$ is a projection onto the subspace spanned by $\hat{q_n}$. This form is known as the projection form of the Spectral Theorem.

Example

Find a 2×2 matrix with eigenvalues $\lambda_1 = 3, \lambda_2 = -2$ and eigenvectors

$$
\begin{aligned}
\vec{v_1} &= \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} -4 \\ 3 \end{bmatrix} \\
\hat{q_1} &= \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \quad \hat{q_2} = \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} \\
A &= \lambda \hat{q_1} \hat{q_1}^T + \lambda_2 \hat{q_2} \hat{q_2}^T \\
&= 3 \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \end{bmatrix} + (-2) \begin{bmatrix} -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix} \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{5} & \frac{12}{5} \\ \frac{12}{5} & \frac{6}{5} \end{bmatrix}
\end{aligned}
$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech