Advanced Linear Algebra

Alvin Lin

January 2019 - May 2019

Orthogonal Bases for Subspaces

Recall:

$$\text{proj}_{\vec{u}}\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\right)\vec{u}$$

Let W be a subspace of $\mathbb{R}^n$ and let $\{\vec{u_1}, \dots, \vec{u_k}\}$ be an orthogonal basis for W. For any vector $\vec{v}$ in $\mathbb{R}^n$, the orthogonal projection of $\vec{v}$ onto W is

$$\text{proj}_W\vec{v} = \left(\frac{\vec{u_1}\cdot\vec{v}}{\vec{u_1}\cdot\vec{u_1}}\right)\vec{u_1} + \cdots + \left(\frac{\vec{u_k}\cdot\vec{v}}{\vec{u_k}\cdot\vec{u_k}}\right)\vec{u_k}$$

The component of $\vec{v}$ orthogonal to W is

$$\text{perp}_W\vec{v} = \vec{v} - \text{proj}_W\vec{v}$$

Example

Let W be the plane $x - y + 2z = 0$ and $\vec{v} = \langle 3, -1, 2\rangle$. Find $\text{proj}_W\vec{v}$ and $\text{perp}_W\vec{v}$ if the basis for W is

$$\vec{u_1} = \begin{bmatrix}1\\1\\0\end{bmatrix} \quad \vec{u_2} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

$$\text{perp}_W\vec{v} = \vec{v} - \text{proj}_W\vec{v} = \begin{bmatrix} \frac{4}{3} \\ -\frac{4}{3} \\ \frac{8}{3} \end{bmatrix}$$

$$\begin{aligned} \text{proj}_W\vec{v} &= \left(\frac{\vec{u_1} \cdot \vec{v}}{\vec{u_1} \cdot \vec{u_1}}\right)\vec{u_1} + \left(\frac{\vec{u_2} \cdot \vec{v}}{\vec{u_2} \cdot \vec{u_2}}\right)\vec{u_2} \\ &= \begin{bmatrix} \frac{5}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix} \end{aligned}$$

Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$ and $\vec{v}$ be a vector in $\mathbb{R}^n$. Then there are unique vectors $\vec{w}$ in W and $\vec{w}^\perp$ in $W^\perp$ such that $\vec{v} = \vec{w} + \vec{w}^\perp$ and $dim(W) + dim(W^T) = n$.

The Gram-Schmidt Process

Given a basis $\{\vec{x_1}, \ldots, \vec{x_k}\}$ for a subspace W, can we construct from it an orthogonal basis? Let $W = span(\vec{x_1}, \vec{x_2})$ where $\vec{x_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\vec{x_2} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$. Construct an orthogonal basis for W. We want to turn $\{\vec{x_1}, \vec{x_2}\}$ into $\{\vec{v_1}, \vec{v_2}\}$ where $\{\vec{x_1}, \vec{x_2}\}$ are not orthogonal and $\{\vec{v_1}, \vec{v_2}\}$ are orthogonal.

$$\begin{aligned} \vec{v_1} &= \vec{x_2} \\ \vec{v_2} &= \text{perp}_{\vec{v_1}}\vec{x_2} \\ &= \vec{x_2} - \text{proj}_{\vec{v_2}}\vec{x_2} \\ &= \vec{x_2} - \left(\frac{\vec{v_1} \cdot \vec{x_2}}{\vec{v_1} \cdot \vec{v_1}}\right)\vec{v_1} \\ &= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{(-2)}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

In general, let $\{\vec{x_1}, \ldots, \vec{x_k}\}$ be a basis for a subspace W of $\mathbb{R}^n$.

$$
\begin{aligned}
\vec{v_1} &= \vec{x_1} \quad W_1 = span(\vec{v_1}) \\
\vec{v_2} &= \vec{x_2} - \text{proj}_{W_1}\vec{x_2} \quad W_2 = span(\vec{v_1}, \vec{v_2}) \\
&= \vec{x_2} - \left(\frac{\vec{v_1} \cdot \vec{x_2}}{\vec{v_1} \cdot \vec{v_1}}\right)\vec{v_1} \\
\vec{v_3} &= \vec{x_3} - \text{proj}_{W_2}\vec{x_3} \quad W_3 = span(\vec{v_1}, \vec{v_2}, \vec{v_3}) \\
&= \vec{x_2} - \left(\frac{\vec{v_1} \cdot \vec{x_3}}{\vec{v_1} \cdot \vec{v_1}}\right)\vec{v_1} - \left(\frac{\vec{v_2} \cdot \vec{x_3}}{\vec{v_2} \cdot \vec{v_2}}\right)\vec{v_2} \\
&\vdots \\
\vec{v_k} &= \vec{x_k} - \text{proj}_{W_{k-1}}\vec{x_k} \quad W_k = span(\vec{v_1}, \ldots, \vec{v_k}) \\
&= \vec{x_k} - \left(\frac{\vec{v_1} \cdot \vec{x_k}}{\vec{v_1} \cdot \vec{v_1}}\right)\vec{v_1} - \cdots - \left(\frac{\vec{v_{k-1}} \cdot \vec{x_k}}{\vec{v_{k-1}} \cdot \vec{v_{k-1}}}\right)\vec{v_{k-1}}
\end{aligned}
$$

Example

Apply the Gram-Schmidt process to construct an orthonormal basis for the subspace.

$$W = span(\vec{x_1}, \vec{x_2}, \vec{x_3}) \text{ of } \mathbb{R}^4 \text{ where}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad \vec{x_2} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{x_3} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\text{Let: } \vec{v_1} = \vec{x_1} \quad W_1 = span(\vec{v_1})$$

$$\begin{aligned} \vec{v_2} &= \text{perp}_{W_1}\vec{x_2} \\ &= \vec{x_2} - \text{proj}_{W_1}\vec{x_2} \\ &= \vec{x_2} - \left(\frac{\vec{v_1} \cdot \vec{x_2}}{\vec{v_1} \cdot \vec{v_1}}\right)\vec{v_1} \\ &= \begin{bmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} \quad W_2 = span(\vec{v_1}, \vec{v_2}) \end{aligned}$$

$$\begin{aligned} \vec{v_3} &= \text{perp}_{W_2}\vec{x_3} \\ &= \vec{x_3} - \text{proj}_{W_3}\vec{x_3} \\ &= \vec{x_3} - \left(\frac{\vec{v_1} \cdot \vec{x_3}}{\vec{v_1} \cdot \vec{v_1}}\right)\vec{v_1} - \left(\frac{\vec{v_2} \cdot \vec{x_3}}{\vec{v_2} \cdot \vec{v_2}}\right)\vec{v_2} \\ &= \begin{bmatrix} -\frac{1}{2} \\ 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} \end{aligned}$$

$$W = span(\vec{v_1}, \vec{v_2}, \vec{v_3}) = span\left(\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}\right)$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech