Advanced Linear Algebra

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January 2019 - May 2019

Orthogonality

Recall that the orthogonal projection of $\vec{v}$ onto $\hat{u}$ is defined as:

$$\begin{aligned}
\vec{v} \cdot \hat{u} &= \|\vec{v}\|\|\hat{u}\| \cos\theta \\
\text{proj}_{\hat{u}}\vec{v} &= (\|\vec{v}\| \cos\theta)\hat{u} \\
&= \|\vec{v}\| \frac{\vec{v} \cdot \hat{u}}{\|\vec{v}\|} \hat{u} \\
&= (\langle v_1, v_2 \rangle \cdot \langle u_1, u_2 \rangle) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= (v_1 u_1 + v_2 u_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \begin{bmatrix} v_1 u_1^2 + v_2 u_1 u_2 \\ v_1 u_1 u_2 + v_2 u_2^2 \end{bmatrix} \\
&= \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \\
&= P\vec{v}
\end{aligned}$$

If we look at the projection matrix, we observe the following:

$$\begin{aligned}
P &= \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix} \\
&= \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \begin{bmatrix} u_1 & u_2 \end{bmatrix} \\
&= \hat{u}\hat{u}^T
\end{aligned}$$

Projection is a linear transformation that can be expressed as a matrix operation.

Example

Given $\hat{u} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $\text{proj}_{\hat{u}}\vec{v}$.

$$
\begin{aligned}
P &= \hat{u}\hat{u}^T \\
&= \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
\text{proj}_{\hat{u}}\vec{v} &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 \\ -4 \end{bmatrix} \\
&= \begin{bmatrix} \frac{7}{2} \\ -\frac{7}{2} \end{bmatrix}
\end{aligned}
$$

Example

Show that $P = \hat{u}\hat{u}^T$ is symmetric.

$$
\begin{aligned}
P^T &= (\hat{u}\hat{u}^T)^T \\
&= (\hat{u}^T)^T(\hat{u})^T \\
&= \hat{u}\hat{u}^T
\end{aligned}
$$

Show that $P^2 = P$ (P is idempotent).

$$
\begin{aligned}
P^2 &= (\hat{u}\hat{u}^T)(\hat{u}\hat{u}^T) \\
&= \hat{u}(\hat{u}^T\hat{u})\hat{u}^T \\
\hat{u}^T\hat{u} &= 1 \\
&= \hat{u}\hat{u}^T
\end{aligned}
$$

Orthogonal Projection of $\vec{v}$ onto a Plane through Origin

Given a plane, its normal vector $\vec{n}$, a vector $\vec{v}$, the distance from the tip of $\vec{v}$ to the plane is some length that can be represented by a vector $-c\vec{n}$. We can represent the projection of $\vec{v}$ onto the plane as $\vec{p} = \vec{v} - c\vec{n}$.

$$\begin{aligned}
\vec{p} &= \vec{v} - c\vec{n} \\
\vec{n} \cdot \vec{p} &= \vec{n} \cdot (\vec{v} - c\vec{n}) \\
0 &= \vec{n} \cdot \vec{v} - c(\vec{n} \cdot \vec{n}) \\
c &= \frac{\vec{n} \cdot \vec{v}}{\vec{n} \cdot \vec{n}}
\end{aligned}$$

Alternatively, if instead of being given a unit vector $\vec{n}$, suppose we are given two orthogonal vectors $\hat{u_1}$ and $\hat{u_2}$ that span the plane. Let $\vec{p_1} = \text{proj}_{\hat{u_1}}\vec{v}$ and $\vec{p_2} = \text{proj}_{\hat{u_2}}\vec{v}$. The projection of $\vec{v}$ onto the plane spanned by $\hat{u_1}$ and $\hat{u_2}$ is equal to $\vec{p} = \vec{p_1} + \vec{p_2}$.

$$\begin{aligned}
\vec{p_1} &= \text{proj}_{\hat{u_1}}\vec{v} = \hat{u_1}\hat{u_1}^T\vec{v} \\
\vec{p_2} &= \text{proj}_{\hat{u_2}}\vec{v} = \hat{u_2}\hat{u_2}^T\vec{v} \\
\vec{p} &= \vec{p_1} + \vec{p_2} \\
&= (\hat{u_1}\hat{u_1}^T + \hat{u_2}\hat{u_2}^T)\vec{v} = P\vec{v}
\end{aligned}$$

Example

Project $\vec{v} = \langle 1, 0, -2 \rangle$ onto the plane $x + y + z = 0$.

$$\begin{aligned}
\langle 1, 1, 1 \rangle \cdot \langle x, y, z \rangle &= 0 \\
\vec{n} &= \langle 1, 1, 1 \rangle \\
c &= \frac{\vec{n} \cdot \vec{v}}{\vec{n} \cdot \vec{n}} \\
&= \frac{1 + 0 - 2}{1 + 1 + 1} \\
&= -\frac{1}{2} \\
\vec{p} &= \vec{v} + \frac{1}{3}\vec{n} \\
&= \langle \frac{2}{3}, -\frac{1}{3}, -\frac{7}{3} \rangle
\end{aligned}$$

Example

Project $\vec{v} = \langle 1, 0, -2 \rangle$ onto $x + y + z = 0$ with $\hat{u_1} = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ and $\hat{u_2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

$$\begin{aligned}
P &= \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{6} & -\frac{2}{6} & -\frac{2}{6} \\ -\frac{2}{6} & \frac{1}{6} & \frac{1}{6} \\ -\frac{2}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \\
&= \frac{1}{6} \begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \\
P\vec{v} &= \frac{1}{6} \begin{bmatrix} 4 & -2 & -2 \\ -2 & 4 & -2 \\ -2 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \\
&= \begin{bmatrix} \frac{8}{6} \\ \frac{2}{6} \\ -\frac{10}{6} \end{bmatrix} \\
&= \begin{bmatrix} \frac{4}{3} \\ \frac{1}{3} \\ -\frac{5}{3} \end{bmatrix}
\end{aligned}$$

Concept

What is the rank of $P = \hat{u}_1 \hat{u}_1{}^T + \hat{u}_2 \hat{u}_2{}^T$?

Every vector in $\mathbb{R}^3$ gets mapped by $P\vec{v}$ onto a two dimensional plane, so that means that the column space of P must be two dimensional.

Orthogonality in $\mathbb{R}^n$

A set of vectors is **orthogonal** if $\vec{v_i} \cdot \vec{v_j} = 0, i \neq j$. For example:

$$\vec{v_1} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

An **orthogonal basis** is a basis that is an orthogonal set.

Example

Find an orthogonal basis for the subspace W of $\mathbb{R}^3$ given by

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x - y + 2z = 0 \right\}$$

$$\begin{bmatrix} y - 2z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$
$$\vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$
$$\vec{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

These two vectors form a basis for W but are not orthogonal. Let $\vec{w} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be orthogonal to $\vec{u}$ in the plane.

$$\begin{aligned} x - y + 2z &= 0 \\ \vec{w} \cdot \vec{u} &= 0 \\ x + y &= 0 \quad y = -x \\ 2x + 2z &= 0 \\ \vec{w} &= \begin{bmatrix} -z \\ z \\ z \end{bmatrix} = z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

This vector $\vec{w}$ satisfies the equation and the orthogonality condition.

$$\vec{w} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{u} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Fact

Let $[\vec{v_1}, \vec{v_2}, \ldots, \vec{v_k}]$ be an orthogonal basis for a subspace W. Let $\vec{w}$ be any vector in W.

$$\begin{aligned}
\vec{w} &= c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_k\vec{v_k} \\
c_1 &= \frac{\vec{v_1} \cdot \vec{w}}{\vec{v_1} \cdot \vec{v_1}} \\
c_2 &= \frac{\vec{v_2} \cdot \vec{w}}{\vec{v_2} \cdot \vec{v_2}} \\
&\cdots \\
c_i &= \frac{\vec{v_i} \cdot \vec{w}}{\vec{v_i} \cdot \vec{v_i}}
\end{aligned}$$

Example

Let $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Find the coordinates of $\vec{w}$ with respect to

$$\vec{v_1} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{v_3} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
\vec{w} &= c_1\vec{v_1} + c_2\vec{v_2} + c_3\vec{v_3} \\
c_1 &= \frac{2+2-3}{4+1+1} = \frac{1}{6} \\
c_2 &= \frac{0+2+3}{0+1+1} = \frac{5}{2} \\
c_3 &= \frac{1-2+3}{1+1+1} = \frac{2}{3}
\end{aligned}$$

Orthonormality

A set of vectors is **orthonormal** if it is an orthogonal set of unit vectors. An orthonormal basis is a basis that is an orthonormal set. Let $[\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_k]$ be an orthonormal basis for a subspace W. Then $\vec{w} = c_1\hat{q}_1 + c_2\hat{q}_2 + \cdots + c_k\hat{q}_k$ where $c_i = \hat{q}_i \cdot \vec{w}$.

Orthogonal Matrices

The columns of an $m \times n$ matrix Q form an orthonormal set if $Q^T Q = I_n$.

$$\begin{bmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \vdots \\ \hat{q}_n \end{bmatrix} \begin{bmatrix} \hat{q}_1 & \hat{q}_2 & \dots & \hat{q}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

Theorem: An $n \times n$ matrix Q whose columns form an orthonormal set is called an "orthogonal matrix". A square matrix Q is orthogonal if and only if $Q^{-1} = Q^T$. Example:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$
$$A^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = A^{-1}$$

Another example:

$$B = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$
$$B^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = B^{-1}$$

The following statements are equivalent:

1. Q is orthogonal.
2. $\|Q\vec{x}\| = \|\vec{x}\|$ for every $\vec{x}$ in $\mathbb{R}^n$.
3. $(Q\vec{x}) \cdot (Q\vec{y}) = \vec{x} \cdot \vec{y}$.

$$\begin{aligned} (Q\vec{x}) \cdot (Q\vec{y}) &= (Q\vec{x})^T (Q\vec{y}) \\ &= \vec{x}^T Q^T Q \vec{y} \\ &= \vec{x}^T \vec{y} \\ &= \vec{x} \cdot \vec{y} \end{aligned}$$

If Q is an orthogonal matrix ($Q^{-1} = Q^T$), the following consequences result:

1. Q^{-1} is orthogonal.

$$\begin{aligned} (Q^{-1})^{-1} &= (Q^{-1})^T \\ Q &= (Q^T)^T \\ Q &= Q \end{aligned}$$

2. $\det(Q) = \pm 1$

3. If λ is an eigenvalue of Q, then $|\lambda| = 1$.

$$\begin{aligned} Q\vec{v} &= \lambda\vec{v} \\ \|Q\vec{v}\| &= \|\lambda\vec{v}\| \\ \|\vec{v}\| &= |\lambda|\|\vec{v}\| \\ 1 &= \lambda \end{aligned}$$

4. If Q_1 and Q_2 are orthogonal then so is Q_1Q_2.

Theorem: Let W be a subspace of $\mathbb{R}^n$. The set of all vectors that are orthogonal to W is called the "orthogonal complement".

$$W^{\perp} = \{\vec{v} \in \mathbb{R}^n : \vec{v} \cdot \vec{w} = 0 \forall \vec{w} \in W\}$$

Theorem: Let A be an $n \times n$ matrix. Then the orthogonal complement of the row space is the null space.

$$(row(A))^T = null(A)$$

$\vec{x}$ is in the null space of A if $\vec{a_i} \cdot \vec{x} = 0$ therefore every vector in the row space is orthogonal to every vector in the null space. Also, note that $(col(A))^{\perp} = null(A^T)$.

Example

Let W be the subspace of $\mathbb{R}^5$ spanned by

$$\vec{w_1} = \begin{bmatrix} 1 \\ -3 \\ 5 \\ 0 \\ 5 \end{bmatrix} \quad \vec{w_2} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ -2 \\ 3 \end{bmatrix} \quad \vec{w_3} = \begin{bmatrix} 0 \\ -1 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

Find a basis for $W^{\perp}$.

$$\begin{bmatrix} 1 & -3 & 5 & 0 & 5 \\ -1 & 1 & 2 & -2 & 3 \\ 0 & -1 & 4 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 5 & 0 & 5 & 0 \\ -1 & 1 & 2 & -2 & 3 & 0 \\ 0 & -1 & 4 & -1 & 5 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 3 & 4 & 0 \\ 0 & 1 & 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -3x_4 - 4x_5 \\ -x_4 - 3x_5 \\ -2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

$$W^{\perp} = span \left(\begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech