Advanced Linear Algebra

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Systems of Linear Ordinary Differential Equations

A solution to an ordinary differential equation is a function that satisfies the differential equation. Suppose we have the following differential equation:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_1 + 2x_2$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = 3x_1 + 2x_2$$

We can convert it into a matrix equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = A\vec{x}$$

If we diagonalize the matrix A, we can obtain a matrix with its eigenvalues:

$$\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = (PDP^{-1})\vec{x}$$
$$P = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$
$$D = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

If we multiply both sides of the equation by P^{-1}, we can group together the term with $\vec{x}$:

$$(P^{-1})\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} = (P^{-1})(PDP^{-1})\vec{x}$$
$$\frac{\mathrm{d}P^{-1}\vec{x}}{\mathrm{d}t} = D(P^{-1}\vec{x})$$

Let $\vec{y} = P^{-1}\vec{x}$, we can now decouple the system of equations and solve it since it is multiplied by a diagonal matrix of eigenvalues.

$$\frac{\mathrm{d}\vec{y}}{\mathrm{d}t} = D\vec{y}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$
$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = 4y_1 \quad y_1 = c_2\mathrm{e}^{4t}$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = -y_2 \quad y_2 = c_2\mathrm{e}^{-t}$$

Using this solution for $\vec{y}$, we can solve for $\vec{x}$:

$$\vec{y} = \begin{bmatrix} c_1\mathrm{e}^{4t} \\ c_2\mathrm{e}^{-t} \end{bmatrix}$$
$$\vec{x} = P\vec{y}$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} c_1\mathrm{e}^{4t} \\ c_2\mathrm{e}^{-t} \end{bmatrix}$$
$$\vec{x} = c_1\mathrm{e}^{4t}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2\mathrm{e}^{-t}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
$$x_1 = 2c_1\mathrm{e}^{4t} - c_2\mathrm{e}^{-t}$$
$$x_2 = 3c_1\mathrm{e}^{4t} - c_2\mathrm{e}^{-t}$$

In general, with coupled systems of linear ordinary differential equations, solutions will be of the form:

$$\vec{x} = c_1\mathrm{e}^{\lambda_1 t}\vec{v_1} + c_2\mathrm{e}^{\lambda_2 t}\vec{v_2} + \cdots + c_n\mathrm{e}^{\lambda_n t}\vec{v_n}$$

Example

Solve the following system of linear ordinary differential equations.

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = x_2 \quad x_1(0) = -5 \quad \frac{\mathrm{d}x_2}{\mathrm{d}t} = -x_1 \quad x_2(0) = 8$$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
\frac{\mathrm{d}\vec{x}}{\mathrm{d}t} &= A\vec{x} \\
\det(A - \lambda I) &= 0 \\
\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} &= 0 \\
\lambda^2 + 1 &= 0 \\
\lambda &= \pm i \\
\vec{v_1} &= \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \vec{v_2} = \begin{bmatrix} 1 \\ -i \end{bmatrix} \\
\vec{x} &= c_1 \mathrm{e}^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} + c_2 \mathrm{e}^{-it}\begin{bmatrix} 1 \\ -i \end{bmatrix} \\
\mathrm{e}^{it} &= \cos(t) + i\sin(t) \\
x_1 &= c_2 \mathrm{e}^{it} + c_2 \mathrm{e}^{-it} \\
&= c_1(\cos(t) + i\sin(t)) + c_2(\cos(t) - i\sin(t)) \\
&= (c_1 + c_2)\cos(t) + (ic_1 - ic_2)\sin(t) \\
x_2 &= c_1 i \mathrm{e}^{it} - ic_2 \mathrm{e}^{-it} \\
&= ic_1(\cos(t) + i\sin(t)) - ic_2(\cos(t) - i\sin(t)) \\
&= (ic_1 - ic_2)\cos(t) - (c_1 + c_2)\sin(t)
\end{aligned}
$$

Since c_1 and c_2 are arbitrary constants, we can substitute them with other arbitrary constants.

$$
\begin{aligned}
d_1 &= c_1 + c_2 \\
d_2 &= ic_1 - ic_2 \\
x_1 &= d_1\cos(t) + d_2\sin(t) \\
x_2 &= d_2\cos(t) - d_1\sin(t)
\end{aligned}
$$

We can use our initial conditions to solve for d_1 and d_2.

$$\begin{aligned}
x_1(0) &= d_1\cos(0) + d_2\sin(0) = -4 \\
d_1 &= -4 \\
x_2(0) &= d_2\cos(0) + d_1\sin(0) = 8 \\
d_2 &= 0 \\
x_1 &= -4\cos(t) + 8\sin(t) \\
x_2 &= 8\cos(t) + 4\sin(t)
\end{aligned}$$

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