

# Advanced Linear Algebra

Alvin Lin

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## Gerschgorin Disks

Let  $A = [a_{ij}]$  be a real or complex  $n \times n$  matrix. Let  $r_i = \sum_{j \neq i} |a_{ij}|$ . The  $i^{\text{th}}$  Gerschgorin disk in the complex plane has center  $a_{ii}$  and radius  $r_i$ .

$$D_i = \{z \in \mathbb{C} \mid \|z - a_{ii}\| = r_i\}$$

**Theorem:** Let  $A$  be an  $n \times n$  matrix. Every eigenvalue of  $A$  is contained within a Gerschgorin disk.

### Example

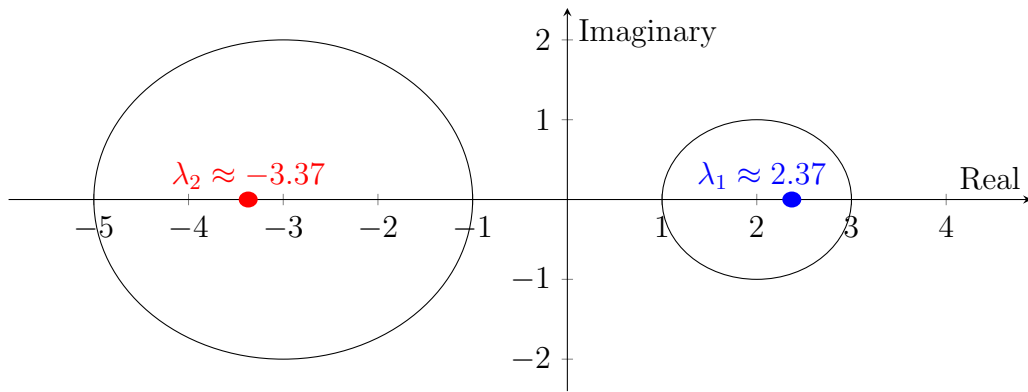
$$A = \begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(2 - \lambda)(-3 - \lambda) - 2 = 0$$

$$\lambda^2 + \lambda - 8 = 0$$

$$\lambda = \frac{-1 \pm \sqrt{1^2 + 32}}{2}$$



**Example**

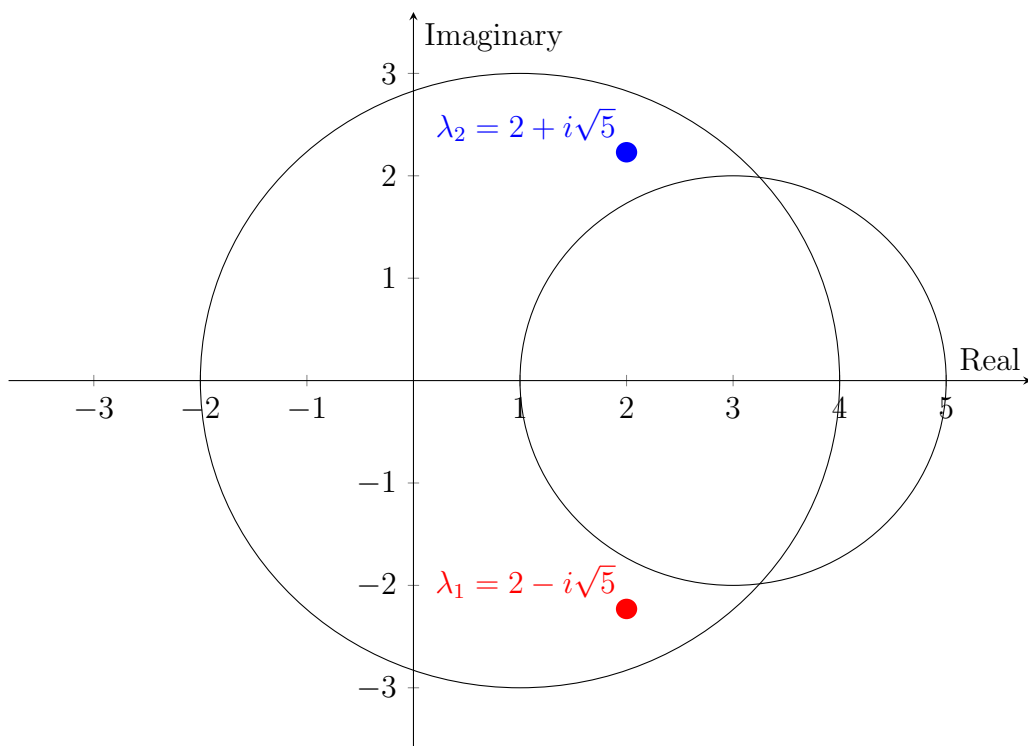
$$A = \begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$(1 - \lambda)(3 - \lambda) + 6 = 0$$

$$\lambda^2 - 4\lambda + 9 = 0$$

$$\begin{aligned} \lambda &= \frac{4 \pm \sqrt{16 - 36}}{2} \\ &= 2 \pm i\sqrt{5} \end{aligned}$$



## Proof

Let  $A\vec{x} = \lambda\vec{x}$ . Let  $x_i$  be the entry of  $\vec{x}$  with the largest absolute value.

$$\begin{aligned}
 [a_{i1} \quad a_{i2} \quad \dots \quad a_{in}] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} &= \begin{bmatrix} \vdots \\ \lambda x_i \\ \vdots \end{bmatrix} \\
 \sum_{j=1}^n a_{ij} x_j &= \lambda x_i \\
 \sum_{j \neq i} a_{ij} x_j &= \lambda x_i - a_{ii} x_i
 \end{aligned}$$

$$\begin{aligned}\lambda - a_{ii} &= \frac{1}{x_i} \sum_{j \neq i} a_{ij} x_j \\ \|\lambda - a_{ii}\| &= \left\| \frac{1}{x_i} \sum_{j \neq i} a_{ij} x_j \right\| \\ &= \frac{1}{\|x_i\|} \left\| \sum_{j \neq i} a_{ij} x_j \right\|\end{aligned}$$

Recall that the Triangle Inequality states:

$$\|z + w\| \leq \|z\| + \|w\|$$

$$\begin{aligned}\|\lambda - a_{ii}\| &\leq \frac{1}{\|x_i\|} \sum_{j \neq i} \|a_{ij}\| \|x_j\| \\ &\leq \sum_{j \neq i} \|a_{ij}\|\end{aligned}$$

This proof summarizes the essence of the theorem of Gerschgorin disks in that all the eigenvalues of a matrix are contained within the Gerschgorin disk radii.

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)