Advanced Linear Algebra

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January 2019 - May 2019

Gerschgorin Disks

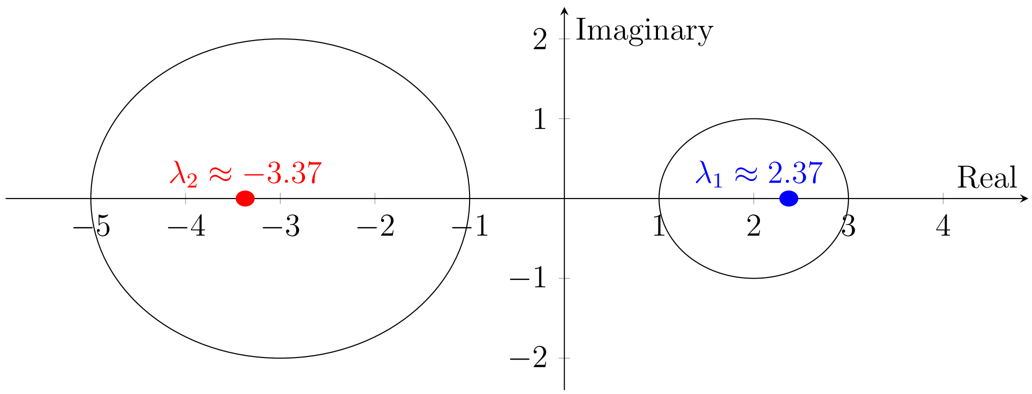
Let $A = [a_{ij}]$ be a real or complex $n \times n$ matrix. Let $r_i = \sum_{j \neq i} |a_{ij}|$. The i^{th} Gerschgorin disk in the complex plane has center a_{ii} and radius r_i.

$$D_i = \{z \in \mathbb{C} \mid \|z - a_{ii}\| = r_i\}$$

Theorem: Let A be an $n \times n$ matrix. Every eigenvalue of A is contained within a Gerschgorin disk.

Example

$$\begin{aligned}
A &= \begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix} \\
\det(A - \lambda I) &= 0 \\
(2 - \lambda)(-3 - \lambda) - 2 &= 0 \\
\lambda^2 + \lambda - 8 &= 0 \\
\lambda &= \frac{-1 \pm \sqrt{1^2 + 32}}{2}
\end{aligned}$$



Example

$$A = \begin{bmatrix} 1 & -3 \\ 2 & 3 \end{bmatrix}$$

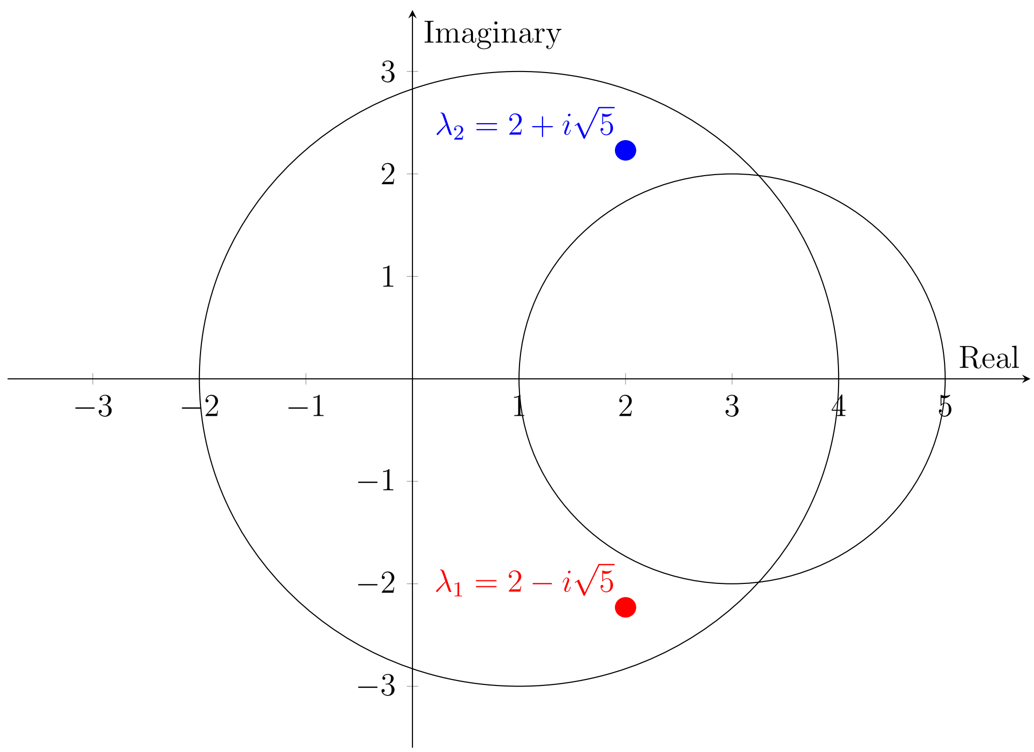
$$\det(A - \lambda I) = 0$$

$$(1-\lambda)(3-\lambda) + 6 = 0$$

$$\lambda^2 - 4\lambda + 9 = 0$$

$$\lambda = \frac{4 + \pm\sqrt{16 - 36}}{2}$$

$$= 2 \pm i\sqrt{5}$$



Proof

Let $A\vec{x} = \lambda\vec{x}$. Let x_i be the entry of $\vec{x}$ with the largest absolute value.

$$\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} \vdots \\ \lambda x_i \\ \vdots \end{bmatrix}$$

$$\sum_{j=1}^{n} a_{ij} x_j = \lambda x_i$$

$$\sum_{j \neq i} a_{ij} x_j = \lambda x_i - a_{ii} x_i$$

$$
\begin{aligned}
\lambda - a_{ii} &= \frac{1}{x_i} \sum_{j \neq i} a_{ij} x_j \\
\|\lambda - a_{ii}\| &= \|\frac{1}{x_i} \sum_{j \neq i} a_{ij} x_j\| \\
&= \frac{1}{\|x_i\|} \|\sum_{j \neq i} a_{ij} x_j\|
\end{aligned}
$$

Recall that the Triangle Inequality states:

$$\|z + w\| \leq \|z\| + \|w\|$$

$$
\begin{aligned}
\|\lambda - a_{ii}\| &\leq \frac{1}{\|x_i\|} \sum_{j \neq i} \|a_{ij}\| \|x_j\| \\
&\leq \sum_{j \neq i} \|a_{ij}\|
\end{aligned}
$$

This proof summarizes the essence of the theorem of Gerschgorin disks in that all the eigenvalues of a matrix are contained with the Gerschgorin disk radii.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech