Advanced Linear Algebra

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The Power Method

This method can find the dominant (largest absolute value) eigenvalue of an $n \times n$ matrix.

Let A be an $n \times n$ diagonalizable matrix with dominant eigenvalue λ. Then there exists a vector $\vec{x_0}$ such that

$$\begin{aligned}
\vec{x_1} &= A\vec{x_0} \\
\vec{x_2} &= A\vec{x_1} = A^2\vec{x_0} \\
\vec{x_3} &= A\vec{x_2} = A^3\vec{x_0} \\
&\vdots \\
\vec{x_k} &= A\overrightarrow{x_{k-1}} = A^k\vec{x_0}
\end{aligned}$$

This series approaches a dominant eigenvector. As a proof, suppose we label the eigenvalues in order.

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_n|$$

with $\vec{v_1}, \ldots, \vec{v_n}$ as eigenvectors. Choose a vector $\vec{x_0} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n}$ (since we know the matrix A is diagonalizable, it must have n linearly independent eigenvectors).

$$\begin{aligned}
A^k\vec{x_0} &= A^k(c_1\vec{v_1} + \cdots + c_n\vec{v_n}) \\
&= c_1A^k\vec{v_1} + \cdots + c_nA^k\vec{v_n} \\
&= c_1\lambda_1^k\vec{v_1} + \cdots + c_n\lambda_n^k\vec{v_n} \\
&= \lambda_1\left[c_1\vec{v_1} + \cdots + c_n(\frac{\lambda_n}{\lambda_1})^k\vec{v_n}\right] \\
\lim_{k\to\infty} A^k\vec{x_0} &= \lambda_1^k c_1\vec{v_1}
\end{aligned}$$

Example

Approximate the dominant eigenvector of $A = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$.

$$\begin{aligned}
\vec{x_0} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\vec{x_1} &= A\vec{x_0} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \\
\vec{x_2} &= \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\
\vec{x_3} &= A\vec{x_2} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} \\
&\vdots \\
\vec{x_7} &= A\vec{x_6} = \begin{bmatrix} 85 \\ 86 \end{bmatrix} \\
\vec{x_8} &= A\vec{x_7} = \begin{bmatrix} 171 \\ 170 \end{bmatrix} \\
\overrightarrow{x_{k+1}} &= A\vec{x_k} \approx \lambda_1 \vec{x_k} \\
\begin{bmatrix} 171 \\ 170 \end{bmatrix} &\approx \lambda_1 \begin{bmatrix} 85 \\ 86 \end{bmatrix} \\
\lambda_1 &\approx 2.01
\end{aligned}$$

The Rayleigh Quotient

The Rayleigh quotient allows us to compute the exact eigenvalue when using the Power method.

$$\begin{aligned}
A\vec{x} &= \lambda_1 \vec{x} \\
(A\vec{x}) \cdot \vec{x} &= \lambda_1 (\vec{x} \cdot \vec{x}) \\
\lambda_1 &= \frac{(A\vec{x}) \cdot \vec{x}}{(\vec{x} \cdot \vec{x})}
\end{aligned}$$

We can check for the convergence of the Power method by computing

$$\lambda_1 \approx \frac{(A\vec{x_k}) \cdot \vec{x_k}}{\vec{x_k} \cdot \vec{x_k}}$$

and determining its change as we iterate to convergence. For real world purposes, we can normalize the vector $\vec{x_k}$ to avoid roundoff error in computers. We can either compute $\hat{x_k} = \frac{\vec{x_k}}{\|\vec{x_k}\|}$ or divide each component of the vector by the largest entry.

The Shifted Power Method

If λ is an eigenvalue of A, then $\lambda - \alpha$ is an eigenvalue of $A - \alpha I$ for any α. So if λ_1 is the dominant eigenvalue of A, then the eigenvalues of $A - \lambda_1 I$ are $0, \lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \ldots, \lambda_n - \lambda_1$. We can apply the Power method to $B = A - \lambda_1 I$ to find $\lambda_2 - \lambda_1$ and hence get λ_2.

The Inverse Power Method

$$A\vec{v} = \lambda\vec{v}$$
$$A^{-1}\vec{v} = \frac{1}{\lambda}\vec{v}$$

We can apply the Power method to A^{-1} to obtain the smallest eigenvalues of A.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech