

# Advanced Linear Algebra

Alvin Lin

January 2019 - May 2019

## Complex Number Review

Suppose we have to find the eigenvalues of the following matrix.

$$A = \begin{bmatrix} 4 & -10 \\ 2 & -4 \end{bmatrix}$$

$$\begin{vmatrix} 4 - \lambda & -10 \\ 2 & -4 - \lambda \end{vmatrix} = 0$$

$$(4 - \lambda)(-4 - \lambda) + 20 = 0$$

$$-16 + \lambda^2 + 20 = 0$$

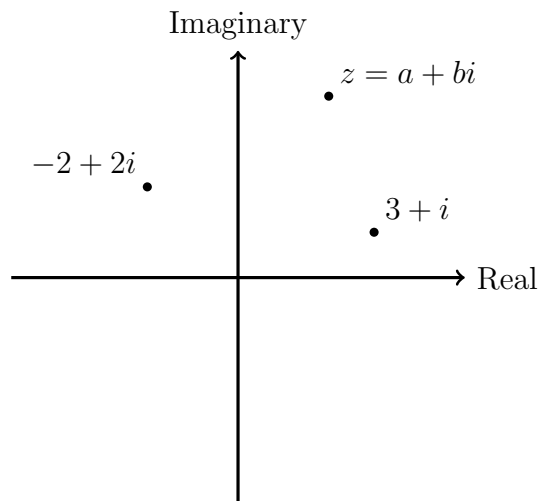
$$\lambda^2 + 4 = 0$$

$$\lambda^2 = -4$$

$$\lambda = 0 \pm 2i$$

## Complex Numbers Review

Complex Plane:



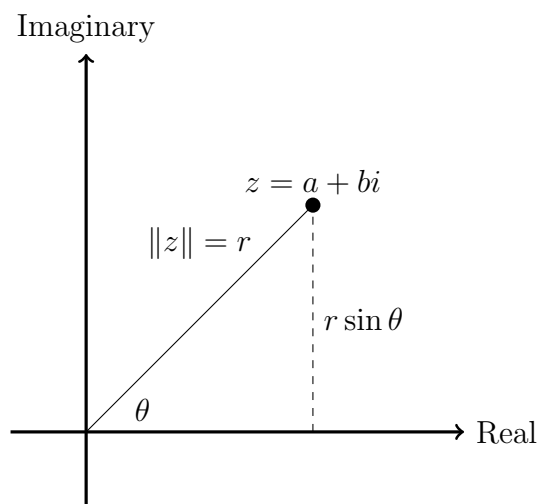
We define  $i^2 = -1$  as a basis for complex numbers. Operations with complex numbers work the same as vectors.

$$\begin{aligned}
 z &= a + bi \\
 w &= c + di \\
 z \pm w &= (a \pm c) + i(b \pm d) \\
 zw &= (a + bi)(c + di) \\
 &= ac + adi + bci + bdi^2 \\
 &= (ac - bd) + i(ad + bc) \\
 \bar{z} &= a - bi \\
 \|z\| &= \sqrt{a^2 + b^2}
 \end{aligned}$$

Example:

$$\begin{aligned}
 z &= -1 + 2i \\
 w &= 3 + 4i \\
 \frac{z}{w} &= \frac{-1 + 2i}{3 + 4i} \frac{3 - 4i}{3 - 4i} \\
 &= \frac{-3 + 4i + 6i - 8i^2}{9 - 12i + 12i - 16i^2} \\
 &= \frac{5 + 10i}{25} \\
 &= \frac{1}{5} + \frac{2}{5}i
 \end{aligned}$$

## Polar Form



$$\begin{aligned}z &= a + bi \\ &= r \cos \theta + ir \sin \theta, \quad -\pi < \theta < \pi\end{aligned}$$

Example:

$$\begin{aligned}z &= 1 + i \\ \theta &= \frac{\pi}{4} \\ 1 + i &= \sqrt{2} \cos\left(\frac{\pi}{4}\right) + i\sqrt{2} \sin\left(\frac{\pi}{4}\right)\end{aligned}$$

Example:

$$\begin{aligned}w &= 1 - \sqrt{3}i \\ \theta &= -\frac{\pi}{3} \quad r = \sqrt{4} = 2 \\ 1 - \sqrt{3}i &= 2 \cos\left(-\frac{\pi}{3}\right) + i2 \sin\left(-\frac{\pi}{3}\right)\end{aligned}$$

Polar form is useful because multiplication of complex numbers is much easier in

polar form.

$$\begin{aligned}z &= r(\cos \theta + i \sin \theta) \\w &= \rho(\cos \phi + i \sin \phi) \\zw &= r\rho [\cos \theta \cos \phi + i \cos \theta \sin \phi + i \sin \theta \cos \phi + i^2 \sin \theta \sin \phi] \\&= r\rho [\cos \theta \cos \phi - \sin \theta \sin \phi + i(\cos \theta \sin \phi + \sin \theta \cos \phi)] \\&= r\rho [\cos(\theta + \phi) + i \sin(\theta + \phi)] \\\frac{z}{w} &= \frac{r}{\rho} [\cos(\theta - \phi) + i \sin(\theta - \phi)]\end{aligned}$$

Example:

$$\begin{aligned}z &= 1 + i \\&= \sqrt{2} \left( \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right) \\w &= i \\&= 1 \left( \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right) \\zw &= \sqrt{2} \left[ \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right] \\&= -1 + i\end{aligned}$$

## DeMoivre's Theorem

Let  $z = r(\cos \theta + i \sin \theta)$ :

$$\begin{aligned}z^n &= r^n(\cos(n\theta) + i \sin(n\theta)) \\z^{\frac{1}{n}} &= r^{\frac{1}{n}} \left[ \cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right) \right] \quad k = 0, 1, \dots, n-1\end{aligned}$$

Example: Find the three cube roots of  $-27 = 27(\cos \pi + i \sin \pi)$ .

$$\begin{aligned}(-27)^{\frac{1}{3}} &= 27^{\frac{1}{3}} \left[ \cos\left(\frac{\pi + 2k\pi}{3}\right) + i \sin\left(\frac{\pi + 2k\pi}{3}\right) \right] \quad k = 0, 1, 2 \\k = 0 : & 3 \left[ \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right] \\k = 1 : & 3 [\cos \pi + i \sin \pi] \\k = 2 : & 3 \left[ \cos \frac{5\pi}{3} + i \sin \frac{5\pi}{3} \right]\end{aligned}$$

You can find all my notes at <http://omgimenerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimenerd.tech](mailto:alvin@omgimenerd.tech)