Advanced Linear Algebra

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Complex Number Review

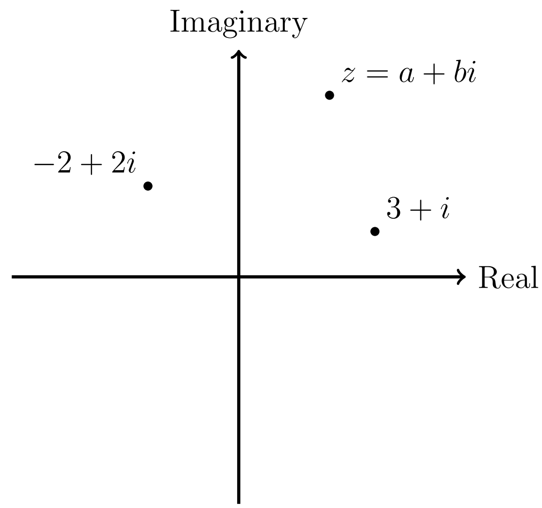
Suppose we have to find the eigenvalues of the following matrix.

$$A = \begin{bmatrix} 4 & -10 \\ 2 & -4 \end{bmatrix}$$

$$\begin{aligned}
\begin{vmatrix} 4-\lambda & -10 \\ 2 & -4-\lambda \end{vmatrix} &= 0 \\
(4-\lambda)(-4-\lambda)+20 &= 0 \\
-16+\lambda^2+20 &= 0 \\
\lambda^2+4 &= 0 \\
\lambda^2 &= -4 \\
\lambda &= 0 \pm 2i
\end{aligned}$$

Complex Numbers Review

Complex Plane:



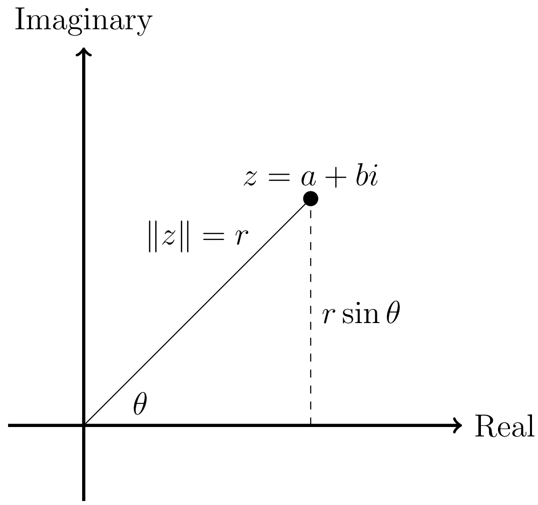
We define $i^2 = -1$ as a basis for complex numbers. Operations with complex numbers work the same as vectors.

$$
\begin{aligned}
z &= a + bi \\
w &= c + di \\
z \pm w &= (a \pm c) + i(b \pm d) \\
zw &= (a + bi)(c + di) \\
&= ac + adi + bci + bdi^2 \\
&= (ac - bd) + i(ad + bc) \\
\bar{z} &= a - bi \\
\|z\| &= \sqrt{a^2 + b^2}
\end{aligned}
$$

Example:

$$
\begin{aligned}
z &= -1 + 2i \\
w &= 3 + 4i \\
\frac{z}{w} &= \frac{-1 + 2i}{3 + 4i} \frac{3 - 4i}{3 - 4i} \\
&= \frac{-3 + 4i + 6i - 8i^2}{9 - 12i + 12i - 16i^2} \\
&= \frac{5 + 10i}{25} \\
&= \frac{1}{5} + \frac{2}{5}i
\end{aligned}
$$

Polar Form



$$
\begin{aligned}
z &= a + bi \\
&= r\cos\theta + ir\sin\theta, \quad -\pi < \theta < \pm
\end{aligned}
$$

Example:

$$
\begin{aligned}
z &= 1 + i \\
\theta &= \frac{\pi}{4} \\
1 + i &= \sqrt{2}\cos(\frac{\pi}{4}) + i\sqrt{2}\sin(\frac{\pi}{4})
\end{aligned}
$$

Example:

$$
\begin{aligned}
w &= 1 - \sqrt{3}i \\
\theta &= -\frac{\pi}{3} \quad r = \sqrt{4} = 2 \\
1 - \sqrt{3}i &= 2\cos(-\frac{\pi}{3}) + i2\sin(-\frac{\pi}{3})
\end{aligned}
$$

Polar form is useful because multiplication of complex numbers is much easier in polar form.

$$\begin{aligned}
z &= r(\cos\theta + i\sin\theta) \\
w &= \rho(\cos\phi + i\sin\phi) \\
zw &= r\rho\left[\cos\theta\cos\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi + i^2\sin\theta\sin\phi\right] \\
&= r\rho\left[\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \sin\theta\cos\phi)\right] \\
&= r\rho\left[\cos(\theta+\phi) + i\sin(\theta+\phi)\right] \\
\frac{z}{w} &= \frac{r}{\rho}\left[\cos(\theta-\phi) + i\sin(\theta-\phi)\right]
\end{aligned}$$

Example:

$$\begin{aligned}
z &= 1 + i \\
&= \sqrt{2}\left(\cos(\frac{\pi}{4}) + i\sin(\frac{\pi}{4})\right) \\
w &= i \\
&= 1\left(\cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2})\right) \\
zw &= \sqrt{2}\left[\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4})\right] \\
&= -1 + i
\end{aligned}$$

DeMoivre's Theorem

Let $z = r(\cos\theta + i\sin\theta)$:

$$\begin{aligned}
z^n &= r^n(\cos(n\theta) + i\sin(n\theta)) \\
z^{\frac{1}{n}} &= r^{\frac{1}{n}}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right] \qquad k = 0, 1, \ldots, n-1
\end{aligned}$$

Example: Find the three cube roots of $-27 = 27(\cos\pi + i\sin\pi)$.

$$\begin{aligned}
(-27)^3 &= 27^{\frac{1}{3}}\left[\cos\left(\frac{\pi + 2k\pi}{3}\right) + i\sin\left(\frac{\pi + 2k\pi}{3}\right)\right] \qquad k = 0, 1, 2 \\
k = 0 &: 3\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right] \\
k = 1 &: 3\left[\cos\pi + i\sin\pi\right] \\
k = 2 &: 3\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right]
\end{aligned}$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech