

Advanced Linear Algebra

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Similarity and Diagonalization

Let A, B be $n \times n$ matrices. A is similar to B ($A \sim B$) if there is an invertible matrix P such that $B = P^{-1}AP$. If $A \sim B$, then:

1. $\det(A) = \det(B)$
2. A and B have the same characteristic polynomial.

We can prove the first statement by taking the determinant the similarity definition.

$$\begin{aligned} B &= P^{-1}AP \\ \det(B) &= \det(P^{-1}AP) \\ &= \det(P^{-1}) \det(A) \det(P) \\ &= \frac{1}{\det(P)} \det(A) \det(P) \\ &= \det(A) \end{aligned}$$

We can prove that similar matrices have the same characteristic polynomial in the same way.

$$\begin{aligned} \det(B - \lambda I) &= \det(P^{-1}AP - \lambda I) \\ &= \det(P^{-1}AP - \lambda P^{-1}IP) \\ &= \det [P^{-1}(A - \lambda I)P] \\ &= \det(A - \lambda I) \end{aligned}$$

Diagonalization

An $n \times n$ matrix A is diagonalizable if there is a diagonal matrix D such that $A \sim D$. An $n \times n$ matrix is diagonalizable if and only if A has n linearly independent eigenvectors.

$$\begin{aligned}P^{-1}AP &= D \\AP &= PD\end{aligned}$$

$$\begin{aligned}A [\vec{p}_1 \ \dots \ \vec{p}_n] &= [\vec{p}_1 \ \dots \ \vec{p}_n] \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \\[A\vec{p}_1 \ \dots \ A\vec{p}_n] &= [\lambda_1\vec{p}_1 \ \dots \ \lambda_n\vec{p}_n]\end{aligned}$$

Example

If possible, find a matrix P that diagonalizes

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

First we need to find the eigenvalues and eigenvectors of A . For brevity, we will skip the steps necessary to do this.

$$\lambda_1 = \lambda_2 = 0 \quad \vec{p} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{p}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \vec{p}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda_3 = -2 \quad \vec{p} = s \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

$$\vec{p}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

$$D = P^{-1}AP$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$$

Example

Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. Compute A^{10} .

$$D = P^{-1}AP$$

$$D^2 = (P^{-1}AP)(P^{-1}AP)$$

$$= P^{-1}A^2P$$

$$D^{10} = P^{-1}A^{10}P$$

$$A^{10} = PD^{10}P^{-1}$$

To use this argument, we need to compute the eigenvalues and eigenvectors of A .

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} -\lambda & 1 \\ 2 & 1 - \lambda \end{vmatrix} = 0$$

$$-\lambda(1 - \lambda) - 2 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda + 1)(\lambda - 2) = 0$$

$$\lambda_1 = -1 \quad \lambda_2 = 2$$

$$\vec{p}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{p}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\begin{aligned} A^{10} &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^{10} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} [(-1)^{10} \quad 0 \quad 0 \quad 2^{10}] \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \\ &= \begin{bmatrix} 342 & 341 \\ 682 & 683 \end{bmatrix} \end{aligned}$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech