Advanced Linear Algebra

Alvin Lin

January 2019 - May 2019

Eigenvalues and Eigenvectors

Let A be an $n \times n$ matrix. λ is an eigenvalue of A if and only if there is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. Such a vector $\vec{x}$ is called an eigenvector.

Example

$$\begin{aligned} A &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ A\vec{x} &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 4 \end{bmatrix} \\ \lambda\vec{x} &= 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Thus, $\lambda = 4$ is an eigenvalue.

Example

Show that $\lambda = 5$ is an eigenvalue of $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

$$\begin{aligned}
A\vec{x} &= 5\vec{x} \\
A\vec{x} - 5\vec{x} &= \vec{0} \\
A\vec{x} - 5I\vec{x} &= \vec{0} \\
(A - 5I)\vec{x} &= \vec{0} \\
\left(\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \vec{x} &= 0 \\
\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{aligned}$$

If we solve this using an augmented matrix and Gauss-Jordan elimination, we get the following:

$$\begin{aligned}
\begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} &\to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
x_1 - \frac{1}{2}x_2 &= 0 \\
x_2 &= s \\
\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}s \\ s \end{bmatrix} = s \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}
\end{aligned}$$

For $\lambda = 5$, $\vec{x} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$, but an eigenvector can be found for any s.

Example

To find eigenvalues, $A\vec{x} = \lambda\vec{x}$ needs to have nontrivial solutions.

$$\begin{aligned}
A\vec{x} - \lambda I\vec{x} &= \vec{0} \\
(A - \lambda I)\vec{x} &= \vec{0}
\end{aligned}$$

In order for this to be true, the determinant of $A - \lambda I$ must be 0. For example:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$det(A - \lambda I) = 0$$

$$det\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

$$det\left(\begin{bmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{bmatrix}\right) = 0$$

$$(3-\lambda)(3-\lambda) - 1 = 0$$

$$9 - 6\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$(\lambda - 4)(\lambda - 2) = 0$$

$$\lambda_1 = 4 \quad \lambda_2 = 2$$

To find eigenvectors for $\lambda_1 = 4$:

$$A\vec{x} = 4\vec{x}$$

$$(A - 4I)\vec{x} = 0$$

$$\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_2 = s$$

$$\vec{x} = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_1 = 4$ has an infinite number of eigenvectors along $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. To find eigenvectors for $\lambda_2 = 2$:

$$A\vec{x} = 2\vec{x}$$

$$(A - 2I)\vec{x} = 0$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_2 = -s$$

$$\vec{x} = \begin{bmatrix} -s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = 2$ has an infinite number of eigenvectors along $\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Example

Find the eigenvalues and eigenvectors of the matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{bmatrix}\right) = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 1 \\ -5 & 4-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 2 & 4-\lambda \end{vmatrix} + 0 = 0$$

$$-\lambda\left[(-\lambda)(4-\lambda) - (-5)\right] - [0-2] = 0$$

$$\lambda^2(4-\lambda) - 5\lambda + 2 = 0$$

$$-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$$

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

$$(\lambda-1)^2(\lambda-2) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = 2$$

Each eigenvalue will have a corresponding set of eigenvectors. For $\lambda = 1$:

$$(A - 1I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - x_3 = 0$$

$$x_2 - x_3 = 0$$

$$x_1 = x_3 = x_2 = s$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For $\lambda = 2$:

$$
\begin{aligned}
(A - 2I)\vec{x} &= \vec{0} \\
\begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 2 & -5 & 2 & 0 \end{bmatrix} &\to \begin{bmatrix} 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
x_1 - \frac{1}{4}x_3 &= 0 \\
x_2 - \frac{1}{2}x_3 &= 0 \\
x_3 &= s \\
\vec{x} &= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{4}s \\ \frac{1}{2}s \\ s \end{bmatrix} = s \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ s \end{bmatrix} s
\end{aligned}
$$

Example

Find the eigenvalues and eigenvectors of the matrix.

$$
\begin{aligned}
A &= \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \\
\det(A - \lambda I) &= 0 \\
\begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} &= 0 \\
\lambda^3 + 2\lambda^2 &= 0 \\
\lambda^2(\lambda + 2) &= 0 \\
\lambda_1 = \lambda_2 &= 0 \\
\lambda_3 &= -2
\end{aligned}
$$

For $\lambda = 0$:

$$\begin{aligned}
(A - 0I)\vec{x} &= \vec{0} \\
\begin{bmatrix} -1 & 0 & 1 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
x_1 &= x_3 \\
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}
\end{aligned}$$

This is a two dimensional eigenspace (the eigenvalue has a geometric multiplicity of 2) spanned by $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Since the eigenvalue $\lambda = 0$ also appears twice, the eigenvalue has a algebraic multiplcity of 2. For $\lambda = -2$:

$$\begin{aligned}
(A + 2I)\vec{x} &= \vec{0} \\
\begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 2 & -3 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
x_1 + x_3 &= 0 \\
x_2 - 3x_3 &= 0 \\
\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} -s \\ 3s \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}
\end{aligned}$$

In summary, the algebraic multiplicity of an eigenvalue is the number of times it appears as a solution to the characteristic polynomial, and the geometric multiplicity of the eigenvalue is the dimensionality of its corresponding eigenspace.

Cayley-Hamilton Theorem

If $\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0 = 0$ is the characteristic equation of A, then $A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = 0$. Essentially, the matrix satisfies its own polynomial. This can be used to prove and verify the following facts.

- $\lambda_1 + \lambda_2 + \cdots + \lambda_n = trace(A)$
  This quantity is also the sum of the diagonal elements.

- $\lambda_1\lambda_2\lambda_3 \dots \lambda_n = \det(A)$
- λ^n is an eigenvalue of A^n.
- $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

Theorem

Suppose an $n \times n$ matrix A has eigenvectors $\vec{v_1}, \dots, \vec{v_m}$ with corresponding eigenvalues $\lambda_1, \dots, \lambda_m$. If $\vec{x} = c_1\vec{v_1} + \cdots + c_m\vec{v_m}$ then $A^k\vec{x} = A^k(c_1\vec{v_1} + \cdots + c_m\vec{v_m}) = c_1\lambda_1^k\vec{v_1} + \cdots + c_m\lambda_m^k\vec{v_m}$.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech