

# Advanced Linear Algebra

Alvin Lin

January 2019 - May 2019

## Eigenvalues and Eigenvectors

Let  $A$  be an  $n \times n$  matrix.  $\lambda$  is an eigenvalue of  $A$  if and only if there is a nonzero vector  $\vec{x}$  such that  $A\vec{x} = \lambda\vec{x}$ . Such a vector  $\vec{x}$  is called an eigenvector.

### Example

$$\begin{aligned} A &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} & \vec{v} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ A\vec{x} &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 4 \end{bmatrix} \\ \lambda\vec{x} &= 4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

Thus,  $\lambda = 4$  is an eigenvalue.

**Example**

Show that  $\lambda = 5$  is an eigenvalue of  $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ .

$$\begin{aligned} A\vec{x} &= 5\vec{x} \\ A\vec{x} - 5\vec{x} &= \vec{0} \\ A\vec{x} - 5I\vec{x} &= \vec{0} \\ (A - 5I)\vec{x} &= \vec{0} \\ \left( \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \right) \vec{x} &= \vec{0} \\ \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

If we solve this using an augmented matrix and Gauss-Jordan elimination, we get the following:

$$\begin{aligned} \begin{bmatrix} -4 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ x_1 - \frac{1}{2}x_2 &= 0 \\ x_2 &= s \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \frac{1}{2}s \\ s \end{bmatrix} = s \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \end{aligned}$$

For  $\lambda = 5$ ,  $\vec{x} = \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ , but an eigenvector can be found for any  $s$ .

**Example**

To find eigenvalues,  $A\vec{x} = \lambda\vec{x}$  needs to have nontrivial solutions.

$$\begin{aligned} A\vec{x} - \lambda I\vec{x} &= \vec{0} \\ (A - \lambda I)\vec{x} &= \vec{0} \end{aligned}$$

In order for this to be true, the determinant of  $A - \lambda I$  must be 0. For example:

$$\begin{aligned}
 A &= \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \\
 \det(A - \lambda I) &= 0 \\
 \det\left(\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) &= 0 \\
 \det\left(\begin{bmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{bmatrix}\right) &= 0 \\
 (3 - \lambda)(3 - \lambda) - 1 &= 0 \\
 9 - 6\lambda + \lambda^2 - 1 &= 0 \\
 \lambda^2 - 6\lambda + 8 &= 0 \\
 (\lambda - 4)(\lambda - 2) &= 0 \\
 \lambda_1 = 4 & \quad \lambda_2 = 2
 \end{aligned}$$

To find eigenvectors for  $\lambda_1 = 4$ :

$$\begin{aligned}
 A\vec{x} &= 4\vec{x} \\
 (A - 4I)\vec{x} &= 0 \\
 \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 x_1 = x_2 = s & \\
 \vec{x} = \begin{bmatrix} s \\ s \end{bmatrix} &= s \begin{bmatrix} 1 \\ 1 \end{bmatrix}
 \end{aligned}$$

$\lambda_1 = 4$  has an infinite number of eigenvectors along  $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ . To find eigenvectors for  $\lambda_2 = 2$ :

$$\begin{aligned}
 A\vec{x} &= 2\vec{x} \\
 (A - 2I)\vec{x} &= 0 \\
 \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
 x_1 = -x_2 = -s & \\
 \vec{x} = \begin{bmatrix} -s \\ s \end{bmatrix} &= s \begin{bmatrix} -1 \\ 1 \end{bmatrix}
 \end{aligned}$$

$\lambda_2 = 2$  has an infinite number of eigenvectors along  $\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ .

### Example

Find the eigenvalues and eigenvectors of the matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\det \left( \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4 - \lambda \end{bmatrix} \right) = 0$$

$$-\lambda \begin{vmatrix} -\lambda & 1 \\ -5 & 4 - \lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 2 & 4 - \lambda \end{vmatrix} + 0 = 0$$

$$-\lambda [(-\lambda)(4 - \lambda) - (-5)] - [0 - 2] = 0$$

$$\lambda^2(4 - \lambda) - 5\lambda + 2 = 0$$

$$-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$$

$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$

$$(\lambda - 1)^2(\lambda - 2) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = 2$$

Each eigenvalue will have a corresponding set of eigenvectors. For  $\lambda = 1$ :

$$(A - 1I)\vec{x} = \vec{0}$$

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - x_3 = 0$$

$$x_2 - x_3 = 0$$

$$x_1 = x_3 = x_2 = s$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For  $\lambda = 2$ :

$$\begin{aligned}(A - 2I)\vec{x} &= \vec{0} \\ \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 2 & -5 & 2 & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{4} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ x_1 - \frac{1}{4}x_3 &= 0 \\ x_2 - \frac{1}{2}x_3 &= 0 \\ x_3 &= s \\ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \begin{bmatrix} \frac{1}{4}s \\ \frac{1}{2}s \\ s \end{bmatrix} = s \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ s \end{bmatrix} s\end{aligned}$$

### Example

Find the eigenvalues and eigenvectors of the matrix.

$$\begin{aligned}A &= \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \\ \det(A - \lambda I) &= 0 \\ \begin{vmatrix} -1 - \lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1 - \lambda \end{vmatrix} &= 0 \\ \lambda^3 + 2\lambda^2 &= 0 \\ \lambda^2(\lambda + 2) &= 0 \\ \lambda_1 = \lambda_2 &= 0 \\ \lambda_3 &= -2\end{aligned}$$

For  $\lambda = 0$ :

$$(A - 0I)\vec{x} = \vec{0}$$
$$\begin{bmatrix} -1 & 0 & 1 & 0 \\ 3 & 0 & -3 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$x_1 = x_3$$
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

This is a two dimensional eigenspace (the eigenvalue has a geometric multiplicity of 2) spanned by  $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ ,  $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ . Since the eigenvalue  $\lambda = 0$  also appears twice, the eigenvalue has an algebraic multiplicity of 2. For  $\lambda = -2$ :

$$(A + 2I)\vec{x} = \vec{0}$$
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 3 & 2 & -3 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$x_1 + x_3 = 0$$
$$x_2 - 3x_3 = 0$$
$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ 3s \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

In summary, the algebraic multiplicity of an eigenvalue is the number of times it appears as a solution to the characteristic polynomial, and the geometric multiplicity of the eigenvalue is the dimensionality of its corresponding eigenspace.

## Cayley-Hamilton Theorem

If  $\lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_1\lambda + c_0 = 0$  is the characteristic equation of  $A$ , then  $A^n + c_{n-1}A^{n-1} + \dots + c_1A + c_0I = 0$ . Essentially, the matrix satisfies its own polynomial. This can be used to prove and verify the following facts.

- $\lambda_1 + \lambda_2 + \dots + \lambda_n = \text{trace}(A)$

This quantity is also the sum of the diagonal elements.

- $\lambda_1 \lambda_2 \lambda_3 \dots \lambda_n = \det(A)$
- $\lambda^n$  is an eigenvalue of  $A^n$ .
- $\frac{1}{\lambda}$  is an eigenvalue of  $A^{-1}$ .

## Theorem

Suppose an  $n \times n$  matrix  $A$  has eigenvectors  $\vec{v}_1, \dots, \vec{v}_m$  with corresponding eigenvalues  $\lambda_1, \dots, \lambda_m$ . If  $\vec{x} = c_1 \vec{v}_1 + \dots + c_m \vec{v}_m$  then  $A^k \vec{x} = A^k (c_1 \vec{v}_1 + \dots + c_m \vec{v}_m) = c_1 \lambda_1^k \vec{v}_1 + \dots + c_m \lambda_m^k \vec{v}_m$ .

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)