

Boundary Value Problems: Homework 12

Alvin Lin

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Problem 1

Find the solution to

$$\begin{aligned}y_{tt} &= a^2 y_{xx} \quad -\infty < x < \infty \quad t > 0 \\y(x, 0) &= 0 \quad -\infty < x < \infty \\y_t(x, 0) &= \cos(x) \quad -\infty < x < \infty \\y(x, t) &= f_1(x + at) + f_2(x - at) \\y(x, 0) &= f_1(x) + f_2(x) = 0 \\f_1(x) &= -f_2(x) \\y_t &= a f_1'(x + at) - a f_2'(x - at) \\y_t(x, 0) &= a f_1'(x) - a f_2'(x) = \cos(x) \\f_1'(x) - f_2'(x) &= \frac{\cos(x)}{a} \\ \int f_1'(x) - f_2'(x) &= \int \frac{\cos(x)}{a} \\f_1(x) - f_2(x) &= \frac{\sin(x)}{a} + c \\f_1(x) - (-f_1(x)) &= \frac{\sin(x)}{a} + c \\f_1(x) &= \frac{\sin(x) + c}{2a} \\f_2(x) &= -\frac{\sin(x) + c}{2a} \\y(x, t) &= \frac{\sin(x + at)}{2a} + \frac{c}{2a} - \frac{\sin(x - at)}{2a} - \frac{c}{2a} \\ &= \frac{\sin(x + at) - \sin(x - at)}{2a}\end{aligned}$$

Problem 2

Find the solution to

$$\begin{aligned}y_{tt} &= a^2 y_{xx} & -\infty < x < \infty & \quad t > 0 \\y(x, 0) &= x^2 & -\infty < x < \infty \\y_t(x, 0) &= 0 & -\infty < x < \infty \\y(x, t) &= f_1(x + at) + f_2(x - at) \\y(x, 0) &= f_1(x) + f_2(x) = x^2 \\y_t(x, t) &= a f_1'(x + at) - a f_2'(x - at) \\y_t(x, 0) &= a f_1'(x) - a f_2'(x) = 0 \\ \int f_1'(x) - f_2'(x) &= \int 0 \\ f_1(x) - f_2(x) &= c \\ f_1(x) + (f_1(x) - c) &= x^2 \\ f_1(x) &= \frac{x^2 + c}{2} \\ f_2(x) &= \frac{x^2 - c}{2} \\ y(x, t) &= \frac{(x + at)^2}{2} + \frac{c}{2} + \frac{(x - at)^2}{2} - \frac{c}{2} \\ &= \frac{(x + at)^2 + (x - at)^2}{2}\end{aligned}$$

Problem 3

Certain solutions of Laplace's equation involve the expressions

$$Y_n(y) = -B_n \tanh\left(\frac{n\pi b}{a}\right) \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right)$$

Here, a and b are positive constants, B_n is an arbitrary constant, and $n \in \mathbb{N}$. Use the definitions of $\sinh(z)$ and $\cosh(z)$ to prove that

$$Y_n(y) = -B_n \tanh\left(\frac{n\pi b}{a}\right) \cosh\left(\frac{n\pi y}{a}\right) + B_n \sinh\left(\frac{n\pi y}{a}\right) = d_n \sinh\left(\frac{n\pi}{a}(y - b)\right)$$

where d_n is a constant.

$$\text{Let : } k = \frac{n\pi}{a}$$

$$\begin{aligned} Y_n(y) &= -B_n \tanh(kb) \cosh(ky) + B_n \sinh(ky) \\ &= -B_n \frac{e^{kb} - e^{-kb}}{e^{kb} + e^{-kb}} \frac{e^{ky} + e^{-ky}}{2} + B_n \frac{e^{ky} - e^{-ky}}{2} \\ &= \frac{B_n}{2} \left(\frac{(e^{ky} - e^{-ky})(e^{kb} + e^{-kb})}{e^{kb} + e^{-kb}} - \frac{(e^{kb} - e^{-kb})(e^{ky} + e^{-ky})}{e^{kb} + e^{-kb}} \right) \\ &= \frac{B_n}{2} \left(\frac{e^{k(y+b)} + e^{k(y-b)} - e^{k(b-y)} - e^{-k(y+b)}}{e^{kb} + e^{-kb}} - \frac{e^{k(b+y)} + e^{k(b-y)} - e^{k(y-b)} - e^{-k(y+b)}}{e^{kb} + e^{-kb}} \right) \\ &= \frac{B_n}{2} \left(\frac{2e^{k(y-b)} - 2e^{k(b-y)}}{e^{kb} + e^{-kb}} \right) \\ &= \frac{2B_n}{e^{kb} + e^{-kb}} \frac{e^{k(y-b)} - e^{-k(y-b)}}{2} \\ &= d_n \sinh(k(y-b)) \\ &= d_n \sinh\left(\frac{n\pi}{a}(y-b)\right) \\ &= \frac{B_n}{\cosh\left(\frac{n\pi b}{a}\right)} \sinh\left(\frac{n\pi}{a}(y-b)\right) \end{aligned}$$

Problem 4

Starting with the solution we went over in class:

$$\begin{aligned} u(x, y) &= E_0(y-b) + \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi(y-b)}{a}\right) \\ E_0 &= -\frac{1}{ab} \int_0^a f(x) dx \\ E_n &= \frac{2}{a \sinh\left(\frac{-n\pi b}{a}\right)} \int_0^a f(x) \cos\left(\frac{n\pi x}{a}\right) dx \quad n = 1, 2, 3, \dots \end{aligned}$$

Solve the boundary value problem. Here $u(x, y)$ represents the steady-state temperature distribution in a rectangular plate, where the edges of the plate at $x = 0$ and $x = a = \pi$ are insulated, the edge at $y = b = 1$

is held fixed at zero temperature, and the temperature along the $y = 0$ edge is given by $4 \cos(6x) + \cos(7x)$.

$$\begin{aligned} \Delta u &= u_{xx} + u_{yy} = 0 \quad 0 < x < \pi \quad 0 < y < 1 \\ u_x(0, y) &= u_x(\pi, y) = 0 \quad 0 < y < 1 \\ u(x, 1) &= 0 \quad 0 < x < \pi \\ u(x, 0) &= 4 \cos(6x) + \cos(7x) \quad 0 < x < \pi \\ E_0 &= -\frac{1}{\pi} \int_0^\pi (4 \cos(6x) + \cos(7x)) dx \\ &= -\frac{1}{\pi} \left[\frac{4 \sin(6x)}{6} - \frac{\sin(7x)}{7} \right]_0^\pi \\ &= 0 \\ E_n &= \frac{2}{\pi \sinh(-n)} \int_0^\pi (4 \cos(6x) + \cos(7x)) \cos(nx) dx \\ &= \frac{2}{\pi \sinh(-n)} \int_0^\pi 2(\cos(6x - nx) + \cos(6x + nx)) + \frac{1}{2}(\cos(7x - nx) + \cos(7x + nx)) dx \\ &= 0? \end{aligned}$$

Problem 5

Starting with separation of variables, find a formal solution to the steady state temperature problem, where three edges of the plate are held fixed at zero temperature, as indicated below.

$$\begin{aligned} \Delta u &= 0 \quad 0 < x < \pi \quad 0 < y < \pi \\ u(0, y) &= u(\pi, y) = 0 \quad 0 < y < \pi \\ u(x, \pi) &= 0 \quad 0 < x < \pi \\ u(x, 0) &= f(x) \quad 0 < x < \pi \end{aligned}$$

$$\begin{aligned} u(x, y) &= X(x)Y(y) \\ X''Y + XY'' &= 0 \\ \frac{X''}{X} &= -\frac{Y''}{Y} = -\alpha^2 \\ X'' + \alpha^2 X &= 0 \\ u(0, y) &= X(0)Y(y) = 0 \quad X(0) = 0 \\ u(\pi, y) &= X(\pi)Y(y) = 0 \quad X(\pi) = 0 \\ u(x, \pi) &= X(x)Y(\pi) = 0 \quad Y(\pi) = 0 \\ r^2 + \alpha^2 &= 0 \\ r &= \pm \alpha i \\ X &= c_1 \cos(\alpha x) + c_2 \sin(\alpha x) \\ X(0) &= 0 = c_1 \cos(0) \\ X(\pi) &= 0 = c_2 \sin(\alpha \pi) \\ \sin(\alpha \pi) &= 0 \\ \alpha \pi &= n\pi \quad \alpha = n \end{aligned}$$

$$\begin{aligned}
Y'' - \alpha^2 Y &= Y'' - n^2 Y = 0 \\
r^2 - n^2 &= 0 \\
r &= \pm n \\
Y &= c_1 e^{ny} + c_2 e^{-ny} \\
&= c_1 \cosh(ny) + c_2 \sinh(-ny) \\
Y(\pi) &= c_1 \cosh(n\pi) + c_2 \sinh(n\pi) = 0 \\
c_1 &= -c_2 \frac{\sinh(n\pi)}{\cosh(n\pi)} = -c_2 \tanh(n\pi) \\
Y_n &= -c_2 \tanh(n\pi) \cosh(ny) + c_2 \sinh(-ny) \\
&= d_n \sinh(n(y - \pi)) \\
&= \frac{c_2}{\cosh(n\pi)} \sinh(n(y - \pi)) \\
u_n(x, y) &= C \frac{\sinh(n(y - \pi))}{\cosh(n\pi)} \sin(nx) \\
u(x, y) &= \sum_{n=1}^{\infty} C_n \frac{\sinh(n(y - \pi))}{\cosh(n\pi)} \sin(nx) \\
u(x, 0) = f(x) &= \sum_{n=1}^{\infty} C_n \frac{\sinh(-n\pi)}{\cosh(n\pi)} \sin(nx) \\
C_n \frac{\sinh(-n\pi)}{\cosh(n\pi)} &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, dx
\end{aligned}$$

Problem 6

Starting with the formula solution you found to the previous problem, solve the following BVP for $u(x, y)$:

$$\begin{aligned}
\Delta u &= 0 \quad 0 < x < \pi \quad 0 < y < \pi \\
u(0, y) &= u(\pi, y) = 0 \quad 0 < y < \pi \\
u(x, \pi) &= 0 \quad 0 < x < \pi \\
u(x, 0) &= \sin(x) + \sin(4x) \quad 0 < x < \pi \\
u(x, y) &= \sum_{n=1}^{\infty} C_n \frac{\sinh(n(y - \pi))}{\cosh(n\pi)} \sin(nx) \\
C_n \frac{\sinh(-n\pi)}{\cosh(n\pi)} &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin(nx) \, dx \\
&= \frac{2}{\pi} \int_0^{\pi} (\sin(x) + \sin(4x)) \sin(nx) \, dx \\
&= 0?
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech