

Boundary Value Problems: Homework 11

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Problem 1

A string is stretched between the points $(0, 0)$ and $(L, 0)$. The initial contour of the string is

$$y(x, 0) = \begin{cases} 0.02x & , 0 \leq x \leq \frac{L}{2} \\ 0.02(L - x) & , \frac{L}{2} \leq x \leq L \end{cases}$$

The string is released from rest. Write an appropriate BVP for the string and solve it. This may be referred to as the *plucked string problem*.

$$y_{tt} = a^2 y_{xx}$$

$$y(0, t) = y(L, t) = 0$$

$$y(x, 0) = \begin{cases} 0.02x & , 0 \leq x \leq \frac{L}{2} \\ 0.02(L - x) & , \frac{L}{2} \leq x \leq L \end{cases}$$

$$y_t(x, 0) = 0$$

$$y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi at}{L}\right) + b_n \sin\left(\frac{n\pi at}{L}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$$

$$y(x, 0) = \sum_{n=1}^{\infty} (a_n \cos(0) + b_n \sin(0)) \sin\left(\frac{n\pi x}{L}\right)$$

$$= \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \begin{cases} 0.02x & , 0 \leq x \leq \frac{L}{2} \\ 0.02(L - x) & , \frac{L}{2} \leq x \leq L \end{cases}$$

$$a_n = \frac{2}{L} \left(\int_0^{\frac{L}{2}} 0.02x \sin\left(\frac{n\pi x}{L}\right) dx + \int_{\frac{L}{2}}^L 0.02(L - x) \sin\left(\frac{n\pi x}{L}\right) dx \right)$$

$$= \frac{0.04}{L} \left(\int_0^{\frac{L}{2}} x \sin\left(\frac{n\pi x}{L}\right) dx + \int_{\frac{L}{2}}^L L \sin\left(\frac{n\pi x}{L}\right) dx - \int_{\frac{L}{2}}^L x \sin\left(\frac{n\pi x}{L}\right) dx \right)$$

$$= \frac{0.04 L^2}{L n\pi} \left(\sin\left(\frac{n\pi}{2}\right) - 1 \right)$$

$$= \frac{0.04L}{n\pi} \left(\sin\left(\frac{n\pi}{2}\right) - 1 \right)$$

$$b_n = 0$$

$$y(x, t) = \frac{0.04L}{n\pi} \sum_{n=1}^{\infty} (\sin(\frac{n\pi}{2}) - 1) \cos(\frac{n\pi at}{L}) \sin(\frac{n\pi x}{L})$$

Problem 2

A string is stretched between $(0, 0)$ and $(2, 0)$ and given an initial velocity

$$y_t(x, 0) = g(x) = \begin{cases} 0.05x & , 0 < x < 1 \\ 0.05(2 - x) & 1 < x < 2 \end{cases}$$

Initially the string is in an equilibrium position along $y = 0$. Construct a BVP matching the given conditions and solve the problem. This may be called the *struck string problem*.

$$\begin{aligned} y_{tt} &= a^2 y_{xx} \\ y(0, t) &= y(2, t) = 0 \\ y(x, 0) &= 0 \\ y_t(x, 0) &= \begin{cases} 0.05x & , 0 < x < 1 \\ 0.05(2 - x) & 1 < x < 2 \end{cases} \\ y(x, t) &= \sum_{n=1}^{\infty} \left(a_n \cos(\frac{n\pi at}{2}) + b_n \sin(\frac{n\pi at}{2}) \right) \sin(\frac{n\pi x}{2}) \\ y(x, 0) &= \sum_{n=1}^{\infty} (a_n \cos(0) + b_n \sin(0)) \sin(\frac{n\pi x}{2}) \\ &= \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{2}) = 0 \\ a_n &= 0 \\ y(x, t) &= \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi at}{2}) \sin(\frac{n\pi x}{2}) \\ y_t(x, t) &= \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{2}) \frac{d}{dt} \left(\sin(\frac{n\pi at}{2}) \right) \\ &= \sum_{n=1}^{\infty} \frac{b_n n\pi a}{2} \cos(\frac{n\pi at}{2}) \sin(\frac{n\pi x}{2}) \\ y_t(x, 0) &= \sum_{n=1}^{\infty} \frac{b_n n\pi a}{2} \sin(\frac{n\pi x}{2}) \\ \frac{b_n n\pi a}{2} &= \frac{2}{2} \left(\int_0^1 0.05x \sin(\frac{n\pi x}{2}) dx + \int_1^2 0.05(2 - x) \sin(\frac{n\pi x}{2}) dx \right) \\ \frac{b_n n\pi a}{0.05} &= \int_0^1 x \sin(\frac{n\pi x}{2}) dx + 2 \int_1^2 \sin(\frac{n\pi x}{2}) dx - \int_1^2 x \sin(\frac{n\pi x}{2}) dx \\ &= \frac{4(\cos(\frac{n\pi}{2}) - 2)}{n\pi} \\ b_n &= \frac{0.8(\cos(\frac{n\pi}{2}) - 2)}{a} \end{aligned}$$

$$y(x, t) = \sum_{n=1}^{\infty} 0.4n\pi \left(\cos\left(\frac{n\pi}{2}\right) - 2 \right) \cos\left(\frac{n\pi at}{2}\right) \sin\left(\frac{n\pi x}{2}\right)$$

Problem 3

Solve the BVP

$$y_{tt} = 4y_{xx} \quad 0 < x < 5 \quad t > 0$$

$$y(0, t) = y(5, t) = 0 \quad t \geq 0$$

$$y(x, 0) = 0$$

$$y_t(x, 0) = \sin(2\pi x) \quad 0 < x < 5$$

$$y(x, t) = \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi 4t}{5}\right) + b_n \sin\left(\frac{n\pi 4t}{5}\right) \right) \sin\left(\frac{n\pi x}{5}\right)$$

$$y(x, 0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{5}\right) = 0$$

$$a_n = 0$$

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{4n\pi t}{5}\right) \sin\left(\frac{n\pi x}{5}\right)$$

$$y_t(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{5}\right) \frac{d}{dt} \left(\sin\left(\frac{4n\pi t}{5}\right) \right)$$

$$= \sum_{n=1}^{\infty} \frac{4b_n n\pi}{5} \cos\left(\frac{4n\pi t}{5}\right) \sin\left(\frac{n\pi x}{5}\right)$$

$$y_t(x, 0) = \sum_{n=1}^{\infty} \frac{4b_n n\pi}{5} \sin\left(\frac{n\pi x}{5}\right) = \sin(2\pi x)$$

$$\frac{4b_n n\pi}{5} = \frac{2}{5} \int_0^5 \sin(2\pi x) \sin\left(\frac{n\pi x}{5}\right) dx$$

$$2b_n n\pi = \int_0^5 \frac{1}{2} \left[\cos\left(2\pi x - \frac{n\pi x}{5}\right) - \cos\left(2\pi x + \frac{n\pi x}{5}\right) \right] dx$$

$$4b_n n\pi = \int_0^5 \cos\left(\frac{(10-n)\pi x}{5}\right) - \cos\left(\frac{(10+n)\pi x}{5}\right) dx$$

$$= 0$$

$$y(x, t) = 0$$

Problem 4

Starting with separation of variables, find a formal solution to

$$\begin{aligned}y_{tt} &= a^2 y_{xx} \quad 0 < x < L \quad x > 0 \\y(0, t) &= y_x(L, t) = 0 \quad t > 0 \\y(x, 0) &= f(x) \quad y_t(x, 0) = 0 \quad 0 < x < L \\y(x, t) &= X(x)T(t) \\XT'' &= a^2 X''T \\ \frac{T''}{a^2 T} &= \frac{X''}{X} = -\lambda \\X'' + \lambda X &= 0\end{aligned}$$

Case $\lambda = 0$

$$\begin{aligned}r^2 &= 0 \quad r = 0 \\X(x) &= c_1 e^0 + c_2 x e^0 = c_1 + x c_2 \\X(0) &= 0 = c_1 \\X'(L) &= c_2 = 0 \\X(x) &= 0\end{aligned}$$

Case $\lambda < 0$

$$\begin{aligned}r^2 - \lambda &= 0 \quad r = \pm\sqrt{\lambda} \\X(x) &= c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x} \\X(0) &= 0 = c_1 + c_2 \\X'(x) &= \sqrt{\lambda}c_1 e^{\sqrt{\lambda}x} - \sqrt{\lambda}c_2 e^{-\sqrt{\lambda}x} \\X'(L) &= \sqrt{\lambda}c_1 e^{\sqrt{\lambda}L} - \sqrt{\lambda}c_2 e^{-\sqrt{\lambda}L} = 0 \\ &= c_1 - c_2 e^{-1} \\c_1 &= c_2 = 0\end{aligned}$$

Case $\lambda > 0$

$$\begin{aligned}r^2 - \lambda &= 0 \quad r = 0 \pm \sqrt{\lambda}i \\X(x) &= c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \\X(0) &= 0 = c_1 \\X'(x) &= \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}x) \\X'(L) &= 0 = \cos(\sqrt{\lambda}L) \\ \sqrt{\lambda}L &= n\pi \\ \lambda &= \left(\frac{n\pi}{L}\right)^2 \\X_n(x) &= c_2 \sin\left(\frac{n\pi x}{L}\right)\end{aligned}$$

$$\begin{aligned}
T'' + a^2\lambda T &= 0 \\
T'' + \left(\frac{an\pi}{L}\right)^2 T &= 0 \\
r^2 &= -\left(\frac{an\pi}{L}\right)^2 \quad r = 0 \pm \frac{an\pi}{L}i \\
T(t) &= c_1 \cos\left(\frac{an\pi t}{L}\right) + c_2 \sin\left(\frac{an\pi t}{L}\right) \\
T'(t) &= -\frac{c_1 an\pi}{L} \sin\left(\frac{an\pi t}{L}\right) + \frac{c_2 an\pi}{L} \cos\left(\frac{an\pi t}{L}\right) \\
T'(0) = 0 &= \frac{c_2 an\pi}{L} \\
c_2 &= 0 \\
T_n(t) &= c_1 \cos\left(\frac{an\pi t}{L}\right) \\
y_n(x, t) &= c \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{an\pi t}{L}\right) \\
y(x, t) &= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{an\pi t}{L}\right) \\
y(x, 0) = f(x) &= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \cos(0) \\
c_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx
\end{aligned}$$

This solution does not match the solution in the back of the textbook, but I could not find any errors in my reasoning.

Problem 5

Starting with the formal solution you found to the previous problem, solve the following boundary value problem for $y(x, t)$:

$$\begin{aligned}
y_{tt} &= a^2 y_{xx} \quad 0 < x < L \quad t > 0 \\
y(0, t) &= y_x(L, t) = 0 \quad t > 0 \\
y(x, 0) &= kx \quad y_t(x, 0) = 0 \quad 0 < x < L \\
y(x, t) &= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{an\pi t}{L}\right) \\
c_n &= \frac{2}{L} \int_0^L kx \sin\left(\frac{n\pi x}{L}\right) dx \\
&= \frac{2k}{L} \left(1 - \frac{L^2}{n\pi}\right) \\
y(x, t) &= \frac{2k}{L} \sum_{n=1}^{\infty} \left(1 - \frac{L^2}{n\pi}\right) \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{an\pi t}{L}\right)
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech