Boundary Value Problems: Homework 11

Alvin Lin

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Problem 1

A string is stretched between the points $(0,0)$ and $(L,0)$. The initial contour of the string is

$$y(x,0) = \begin{cases} 0.02x & ,0 \le x \le \frac{L}{2} \\ 0.02(L-x) & ,\frac{L}{2} \le x \le L \end{cases}$$

The string is released from rest. Write an appropriate BVP for the string and solve it. This may be referred to as the *plucked string problem*.

$$\begin{aligned}
y_{tt} &= a^2 y_{xx} \\
y(0,t) &= y(L,t) = 0 \\
y(x,0) &= \begin{cases} 0.02x & ,0 \le x \le \frac{L}{2} \\ 0.02(L-x) & ,\frac{L}{2} \le x \le L \end{cases} \\
y_t(x,0) &= 0 \\
y(x,t) &= \sum_{n=1}^{\infty} \left(a_n \cos(\frac{n\pi a t}{L}) + b_n \sin(\frac{n\pi a t}{L}) \right) \sin(\frac{n\pi x}{L}) \\
y(x,0) &= \sum_{n=1}^{\infty} (a_n \cos(0) + b_n \sin(0)) \sin(\frac{n\pi x}{L}) \\
&= \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{L}) \\
&= \begin{cases} 0.02x & ,0 \le x \le \frac{L}{2} \\ 0.02(L-x) & ,\frac{L}{2} \le x \le L \end{cases} \\
a_n &= \frac{2}{L} \left(\int_0^{\frac{L}{2}} 0.02x \sin(\frac{n\pi x}{L})\ \mathrm{d}x + \int_{\frac{L}{2}}^{L} 0.02(L-x) \sin(\frac{n\pi x}{L})\ \mathrm{d}x \right) \\
&= \frac{0.04}{L} \left(\int_0^{\frac{L}{2}} x \sin(\frac{n\pi x}{L})\ \mathrm{d}x + \int_{\frac{L}{2}}^{L} L \sin(\frac{n\pi x}{L})\ \mathrm{d}x - \int_{\frac{L}{2}}^{L} x \sin(\frac{n\pi x}{L})\ \mathrm{d}x \right) \\
&= \frac{0.04}{L} \frac{L^2}{n\pi} \left(\sin(\frac{n\pi}{2}) - 1 \right) \\
&= \frac{0.04L}{n\pi} \left(\sin(\frac{n\pi}{2}) - 1 \right) \\
b_n &= 0
\end{aligned}$$

$$y(x,t) = \frac{0.04L}{n\pi}\sum_{n=1}^{\infty}(\sin(\frac{n\pi}{2}) - 1)\cos(\frac{n\pi at}{L})\sin(\frac{n\pi x}{L})$$

Problem 2

A string is stretched between $(0,0)$ and $(2,0)$ and given an initial velocity

$$y_t(x,0) = g(x) = \begin{cases} 0.05x & ,0 < x < 1 \\ 0.05(2-x) & 1 < x < 2 \end{cases}$$

Initially the string is in an equilibrium position along $y = 0$. Construct a BVP matching the given conditions and solve the problem. This may be called the *struck string problem*.

$$\begin{aligned}
y_{tt} &= a^2 y_{xx} \\
y(0,t) &= y(2,t) = 0 \\
y(x,0) &= 0 \\
y_t(x,0) &= \begin{cases} 0.05x & ,0 < x < 1 \\ 0.05(2-x) & 1 < x < 2 \end{cases} \\
y(x,t) &= \sum_{n=1}^{\infty}\left(a_n\cos(\frac{n\pi at}{2}) + b_n\sin(\frac{n\pi at}{2})\right)\sin(\frac{n\pi x}{2}) \\
y(x,0) &= \sum_{n=1}^{\infty}(a_n\cos(0) + b_n\sin(0))\sin(\frac{n\pi x}{2}) \\
&= \sum_{n=1}^{\infty} a_n\sin(\frac{n\pi x}{2}) = 0 \\
a_n &= 0 \\
y(x,t) &= \sum_{n=1}^{\infty} b_n\sin(\frac{n\pi at}{2})\sin(\frac{n\pi x}{2}) \\
y_t(x,t) &= \sum_{n=1}^{\infty} b_n\sin(\frac{n\pi x}{2})\frac{\mathrm{d}}{\mathrm{d}t}\left(\sin(\frac{n\pi at}{2})\right) \\
&= \sum_{n=1}^{\infty}\frac{b_n n\pi a}{2}\cos(\frac{n\pi at}{2})\sin(\frac{n\pi x}{2}) \\
y_t(x,0) &= \sum_{n=1}^{\infty}\frac{b_n n\pi a}{2}\sin(\frac{n\pi x}{2}) \\
\frac{b_n n\pi a}{2} &= \frac{2}{2}\left(\int_0^1 0.05x\sin(\frac{n\pi x}{2})\,\mathrm{d}x + \int_1^2 0.05(2-x)\sin(\frac{n\pi x}{2})\,\mathrm{d}x\right) \\
\frac{b_n n\pi a}{0.05} &= \int_0^1 x\sin(\frac{n\pi x}{2})\,\mathrm{d}x + 2\int_1^2 \sin(\frac{n\pi x}{2})\,\mathrm{d}x - \int_1^2 x\sin(\frac{n\pi x}{2})\,\mathrm{d}x \\
&= \frac{4(cos(\frac{n\pi}{2}) - 2)}{n\pi} \\
b_n &= \frac{0.8(\cos(\frac{n\pi}{2}) - 2)}{a}
\end{aligned}$$

$$y(x,t) = \sum_{n=1}^{\infty} 0.4n\pi(\cos(\frac{n\pi}{2}) - 2)\cos(\frac{n\pi at}{2})\sin(\frac{n\pi x}{2})$$

Problem 3

Solve the BVP

$$\begin{aligned}
y_{tt} &= 4y_{xx} \quad 0 < x < 5 \quad t > 0 \\
y(0,t) &= y(5,t) = 0 \quad t \geq 0 \\
y(x,0) &= 0 \\
y_t(x,0) &= \sin(2\pi x) \quad 0 < x < 5 \\
y(x,t) &= \sum_{n=1}^{\infty} \left(a_n \cos(\frac{n\pi 4t}{5}) + b_n \sin(\frac{n\pi 4t}{5})\right) \sin(\frac{n\pi x}{5}) \\
y(x,0) &= \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{5}) = 0 \\
a_n &= 0 \\
y(x,t) &= \sum_{n=1}^{\infty} b_n \sin(\frac{4n\pi t}{5}) \sin(\frac{n\pi x}{5}) \\
y_t(x,t) &= \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{5}) \frac{\mathrm{d}}{\mathrm{d}t}\left(\sin(\frac{4n\pi t}{5})\right) \\
&= \sum_{n=1}^{\infty} \frac{4b_n n\pi}{5} \cos(\frac{4n\pi t}{5}) \sin(\frac{n\pi x}{5}) \\
y_t(x,0) &= \sum_{n=1}^{\infty} \frac{4b_n n\pi}{5} \sin(\frac{n\pi x}{5}) = \sin(2\pi x) \\
\frac{4b_n n\pi}{5} &= \frac{2}{5}\int_0^5 \sin(2\pi x)\sin(\frac{n\pi x}{5})\ \mathrm{d}x \\
2b_n n\pi &= \int_0^5 \frac{1}{2}\left[\cos(2\pi x - \frac{n\pi x}{5}) - \cos(2\pi x + \frac{n\pi x}{5})\right]\ \mathrm{d}x \\
4b_n n\pi &= \int_0^5 \cos(\frac{(10-n)\pi x}{5}) - \cos(\frac{(10+n)\pi x}{5})\ \mathrm{d}x \\
&= 0 \\
y(x,t) &= 0
\end{aligned}$$

Problem 4

Starting with separation of variables, find a formal solution to

$$
\begin{aligned}
y_{tt} &= a^2 y_{xx} \quad 0 < x < L \quad x > 0 \\
y(0,t) &= y_x(L,t) = 0 \quad t > 0 \\
y(x,0) &= f(x) \quad y_t(x,0) = 0 \quad 0 < x < L \\
y(x,t) &= X(x)T(t) \\
XT'' &= a^2 X'' T \\
\frac{T''}{a^2 T} &= \frac{X''}{X} = -\lambda \\
X'' + \lambda X &= 0 \\
\text{Case } \lambda &= 0 \\
r^2 &= 0 \quad r = 0 \\
X(x) &= c_1 \mathrm{e}^0 + c_2 x \mathrm{e}^0 = c_1 + x c_2 \\
X(0) &= 0 = c_1 \\
X'(L) &= c_2 = 0 \\
X(x) &= 0 \\
\text{Case } \lambda &< 0 \\
r^2 - \lambda &= 0 \quad r = \pm\sqrt{\lambda} \\
X(x) &= c_1 \mathrm{e}^{\sqrt{\lambda}x} + c_2 \mathrm{e}^{-\sqrt{\lambda}x} \\
X(0) &= 0 = c_1 + c_2 \\
X'(x) &= \sqrt{\lambda} c_1 \mathrm{e}^{\sqrt{\lambda}x} - \sqrt{\lambda} c_2 \mathrm{e}^{-\sqrt{\lambda}x} \\
X'(L) &= \sqrt{\lambda} c_1 \mathrm{e}^{\sqrt{\lambda}L} - \sqrt{\lambda} c_2 \mathrm{e}^{-\sqrt{\lambda}L} = 0 \\
&= c_1 - c_2 \mathrm{e}^{-1} \\
c_1 &= c_2 = 0 \\
\text{Case } \lambda &> 0 \\
r^2 - \lambda &= 0 \quad r = 0 \pm \sqrt{\lambda} i \\
X(x) &= c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \\
X(0) &= 0 = c_1 \\
X'(x) &= \sqrt{\lambda} c_2 \cos(\sqrt{\lambda}x) \\
X'(L) &= 0 = \cos(\sqrt{\lambda}L) \\
\sqrt{\lambda}L &= n\pi \\
\lambda &= (\frac{n\pi}{L})^2 \\
X_n(x) &= c_2 \sin(\frac{n\pi x}{L})
\end{aligned}
$$

$$\begin{aligned}
T'' + a^2\lambda T &= 0 \\
T'' + (\frac{an\pi}{L})^2 T &= 0 \\
r^2 &= -(\frac{an\pi}{L})^2 \quad r = 0 \pm \frac{an\pi}{L} i \\
T(t) &= c_1 \cos(\frac{an\pi t}{L}) + c_2 \sin(\frac{an\pi t}{L}) \\
T'(t) &= -\frac{c_1 an\pi}{L} \sin(\frac{an\pi t}{L}) + \frac{c_2 an\pi}{L} \cos(\frac{an\pi t}{L}) \\
T'(0) &= 0 = \frac{c_2 an\pi}{L} \\
c_2 &= 0 \\
T_n(t) &= c_1 \cos(\frac{an\pi t}{L}) \\
y_n(x,t) &= c \sin(\frac{n\pi x}{L}) \cos(\frac{an\pi t}{L}) \\
y(x,t) &= \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L}) \cos(\frac{an\pi t}{L}) \\
y(x,0) &= f(x) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L}) \cos(0) \\
c_n &= \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) \, \mathrm{d}x
\end{aligned}$$

This solution does not match the solution in the back of the textbook, but I could not find any errors in my reasoning.

Problem 5

Starting with the formal solution you found to the previous problem, solve the following boundary value problem for $y(x,t)$:

$$\begin{aligned}
y_{tt} &= a^2 y_{xx} \quad 0 < x < L \quad t > 0 \\
y(0,t) &= y_x(L,t) = 0 \quad t > 0 \\
y(x,0) &= kx \quad y_t(x,0) = 0 \quad 0 < x < L \\
y(x,t) &= \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L}) \cos(\frac{an\pi t}{L}) \\
c_n &= \frac{2}{L} \int_0^L kx \sin(\frac{n\pi x}{L}) \, \mathrm{d}x \\
&= \frac{2k}{L}(1 - \frac{L^2}{n\pi}) \\
y(x,t) &= \frac{2k}{L} \sum_{n=1}^{\infty} (1 - \frac{L^2}{n\pi}) \sin(\frac{n\pi x}{L}) \cos(\frac{an\pi t}{L})
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech