Boundary Value Problems: Homework 10

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Problem 1

Classify the following boundary conditions as Dirichlet, Neumann, separated, or periodic. If the boundary conditions are Dirichlet, Neumann, or separated, also state whether the conditions are homogeneous or nonhomogeneous.

(a) $y'(1) = 0, y'(2) = 1$
Nonhomogeneous Neumann

(b) $u(0,t) = u(9,t), u_x(0,t) = u_x(9,t)$
Periodic

(c) $u(\pi, y) = 0, u(\pi, y) = 0$
Homogeneous Dirichlet

(d) $3y(0) - 2y'(0) = 6, y(5) + 4y'(5) = 8$
Nonhomogeneous separated

Problem 2

The ends of a rod are at temperatures 10°C and 50°C. the rod is 4 units long and has an initial temperature distribution of 30°C. The diffusivity constant a^2 is 4. The corresponding boundary value problem is then

$$\begin{aligned} u_t &= 4u_{xx} \quad 0 < x < 4, t > 0 \\ u(0,t) &= 10 \quad u(4,t) = 50, t \geq 0 \\ u(x,0) &= 30 \quad 0 < x < 4 \end{aligned}$$

Find the temperature $u(x,t)$.

$$\begin{aligned} u(x,t) &= \psi(x) + u(x,t) \\ \psi_t(x) + v_t(x,t) &= 4(\psi_{xx}(x) + v_{xx}(x,t)) \\ v_t(x,t) &= 4(\psi_{xx}(x) + v_{xx}(x,t)) \\ u(0,t) &= \psi(0) + v(0,t) = 10 \\ u(4,t) &= \psi(4) + v(4,t) = 50 \\ t &\to \infty \\ 0 &= 4\psi_{xx}(x) \\ \psi(0) &= 10 \\ \psi(4) &= 50 \end{aligned}$$

$$
\begin{aligned}
\psi(x) &= Ax + B \\
\psi(0) &= 10 = B \\
\psi(4) &= A(4) + 10 = 50 \\
A &= 10 \\
\psi(x) &= 10x + 10 \\
u(0,t) &= \psi(0) + v(0,t) = 10 \\
u(4,t) &= \psi(4) + v(4,t) = 50 \\
v(0,t) &= v(4,t) = 0 \\
u(x,0) &= \psi(x) + v(x,0) = 30 \\
v(x,0) &= 30 - \psi(x) \\
&= 30 - (10x + 10) \\
&= 20 - 10x \\
v(x,t) &= \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{4}) \mathrm{e}^{-4(\frac{n\pi}{4})^2 t} \\
b_n &= \frac{2}{4} \int_0^4 (20 - 10x) \sin(\frac{n\pi x}{4})\ \mathrm{d}x \\
&= \frac{40 + 40\cos(n\pi)}{n\pi} \\
u(x,t) &= \psi(x) + v(x,t) \\
&= 10x + 10 + \sum_{n=1}^{\infty} \frac{40 + 40\cos(n\pi)}{n\pi} \sin(\frac{n\pi x}{4}) \mathrm{e}^{-4(\frac{n\pi}{4})^2 t}
\end{aligned}
$$

Problem 3

Solve the boundary value problem

$$
\begin{aligned}
u_t = u_{xx} &\quad 0 < x < \pi, t > 0 \\
u(0,t) = 0 \quad u(\pi,t) = T_0 &\quad t \geq 0 \\
u(x,0) = 0 &\quad 0 < x < \pi
\end{aligned}
$$

$$
\begin{aligned}
u(x,t) &= \psi(x) + v(x,t) \\
\psi_t + v_t &= \psi_{xx} + v_{xx} \\
u(0,t) &= \psi(0) + v(0,t) = 0 \\
u(\pi,t) &= \psi(\pi) + v(\pi,t) = T_0 \\
t &\to \infty \\
0 &= \psi_{xx} \\
\psi(0) &= 0 \\
\psi(\pi) &= T_0 \\
\psi(x) &= Ax + B \\
\psi(0) &= B = 0 \\
\psi(\pi) &= A(\pi) = T_0 \\
\psi(x) &= \frac{T_0 x}{\pi}
\end{aligned}
$$

$$
\begin{aligned}
v_t &= \psi_{xx} + v_{xx} \\
u(0,t) &= \psi(0) + v(0,t) = 0 \\
u(\pi,t) &= \psi(\pi) + v(\pi,t) = T_0 \\
v(0,t) &= v(\pi,t) = 0 \\
u(x,0) &= \psi(x) + v(x,0) = 0 \\
v(x,0) &= -\psi(x) = -\frac{T_0 x}{\pi} \\
v(x,t) &= \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{\pi}) \mathrm{e}^{-(\frac{n\pi}{\pi})^2 t} \\
b_n &= \frac{2}{\pi} \int_0^{\pi} (-\frac{T_0 x}{\pi}) \sin(\frac{n\pi x}{\pi}) \ \mathrm{d}x \\
&= \frac{2T_0 \cos(n\pi)}{n\pi} \\
u(x,t) &= \frac{T_0 x}{\pi} + \sum_{n=1}^{\infty} \frac{2T_0 \cos(n\pi)}{n\pi} \sin(nx) \mathrm{e}^{-n^2 t}
\end{aligned}
$$

Problem 4

Given the boundary value problem

$$
\begin{aligned}
u_t &= u_{xx} + k\sin(x) \quad 0 < x < \pi, t > 0 \\
u(0,t) &= u(\pi,t) = 0 \quad t \geq 0 \\
u(x,0) &= \sin(x) \quad 0 \leq x \leq \pi
\end{aligned}
$$

Determine $u(x,t)$.

$$
\begin{aligned}
u(x,t) &= \psi(x) + v(x,t) \\
\psi_t + v_t &= \psi_{xx} + v_{xx} + k\sin(x) \\
v_t &= \psi_{xx} + v_{xx} + k\sin(x) \\
u(0,t) &= \psi(0) + v(0,t) = 0 \\
u(\pi,t) &= \psi(\pi) + v(\pi,t) = 0 \\
t &\to \infty \\
0 &= \psi_{xx} + k\sin(x) \\
\psi(0) &= \psi(\pi) = 0 \\
\psi_{xx}(x) &= -k\sin(x) \\
\psi_x(x) &= k\cos(x) + A \\
\psi(x) &= k\sin(x) + Ax + B \\
\psi(0) &= k\sin(0) + 0 + B = 0 \\
\psi(\pi) &= k\sin(\pi) + A\pi + 0 = 0 \\
A &= 0 \quad B = 0 \\
\psi(x) &= k\sin(x) \\
u(0,t) &= \psi(0) + v(0,t) = 0 \\
u(\pi,t) &= \psi(\pi) + v(\pi,t) = 0 \\
v(0,t) &= v(\pi,t) = 0 \\
u(x,0) &= \psi(x) + v(x,0) = \sin(x) \\
v(x,0) &= \sin(x) - k\sin(x) = (1-k)\sin(x) \\
v(x,t) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right) \mathrm{e}^{-(\frac{n\pi x}{\pi})^2 t} \\
b_n &= \frac{2}{\pi}\int_0^{\pi} (1-k)\sin(x)\sin\left(\frac{n\pi x}{\pi}\right)\,\mathrm{d}x = 0 \\
v(x,t) &= 0 \\
u(x,t) &= k\sin(x)
\end{aligned}
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If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech