

Boundary Value Problems: Homework 9

Alvin Lin

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Problem 1

Find all eigenvalues and eigenfunctions of the following boundary value problem:

$$y'' + \lambda y = 0, \quad y(0) = y'(2) = 0$$

Case $\lambda > 0$

$$r^2 + \lambda = 0$$

$$r = 0 \pm \sqrt{\lambda}i$$

$$y = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

$$y(0) = 0 = c_1$$

$$y' = \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}x)$$

$$y'(2) = 0 = \sqrt{\lambda}c_2 \cos(2\sqrt{\lambda})$$

$$0 = \cos(2\sqrt{\lambda})$$

$$2\sqrt{\lambda} = (2n - 1)\frac{\pi}{2}$$

$$\lambda_n = \left(\frac{\pi(2n - 1)}{4}\right)^2$$

$$y_n = c_2 \sin\left(\frac{\pi(2n - 1)x}{4}\right)$$

Case $\lambda = 0$

$$r^2 = 0$$

$$r = 0$$

$$y = c_1 e^0 + c_2 x e^0$$

$$= c_1 + x c_2$$

$$y(0) = 0 = c_1$$

$$y' = c_2$$

$$y'(2) = 0 = c_2$$

$$y = 0$$

Case $\lambda < 0$

$$r^2 - \lambda = 0$$

$$r = \pm\sqrt{\lambda}$$

$$y = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

$$y(0) = 0 = c_1 + c_2$$

$$y' = \sqrt{\lambda}c_1 e^{\sqrt{\lambda}x} - \sqrt{\lambda}c_2 e^{-\sqrt{\lambda}x}$$

$$\begin{aligned} y'(2) = 0 &= \sqrt{\lambda}c_1 e^{2\sqrt{\lambda}x} + \sqrt{\lambda}c_2 e^{-2\sqrt{\lambda}x} \\ &= e^{2\sqrt{\lambda}x} + e^{-2\sqrt{\lambda}x} \end{aligned}$$

$$c_1 = c_2 = 0$$

Problem 2

Show that $y = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$ can be rewritten as $y = d_1 \cosh(\sqrt{-\lambda}x) + d_2 \sinh(\sqrt{-\lambda}x)$ by choosing the values of constants d_1 and d_2 appropriately. Assume c_1 and c_2 are constants and that $\lambda < 0$.

$$\begin{aligned} y &= c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} \\ \text{Let : } c_1 &= \frac{d_1 + d_2}{2} \quad c_2 = \frac{d_1 - d_2}{2} \\ &= \left(\frac{d_1 + d_2}{2}\right)e^{\sqrt{-\lambda}x} + \left(\frac{d_1 - d_2}{2}\right)e^{-\sqrt{-\lambda}x} \\ &= \frac{d_1}{2}e^{\sqrt{-\lambda}x} + \frac{d_2}{2}e^{\sqrt{-\lambda}x} + \frac{d_1}{2}e^{-\sqrt{-\lambda}x} - \frac{d_2}{2}e^{-\sqrt{-\lambda}x} \\ &= \frac{d_1}{2}e^{\sqrt{-\lambda}x} + \frac{d_1}{2}e^{-\sqrt{-\lambda}x} + \frac{d_2}{2}e^{\sqrt{-\lambda}x} - \frac{d_2}{2}e^{-\sqrt{-\lambda}x} \\ &= d_1 \left(\frac{e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}}{2}\right) + d_2 \left(\frac{e^{\sqrt{-\lambda}x} - e^{-\sqrt{-\lambda}x}}{2}\right) \\ &= d_1 \cosh(\sqrt{-\lambda}x) + d_2 \sinh(\sqrt{-\lambda}x) \end{aligned}$$

Problem 3

Find all eigenvalues and eigenfunctions of the following boundary value problem:

$$y'' + \lambda y = 0 \quad y(0) = y(1) \quad y'(0) = y'(1)$$

Case $\lambda > 0$

$$r^2 + \lambda = 0$$

$$r = 0 \pm \sqrt{\lambda}i$$

$$y = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

$$c_1 \cos(0) + c_2 \sin(0) = c_1 \cos(\sqrt{\lambda}) + c_2 \sin(\sqrt{\lambda})$$

$$c_1 = c_1 \cos(\sqrt{\lambda}) + c_2 \sin(\sqrt{\lambda})$$

$$c_1 - c_1 \cos(\sqrt{\lambda}) = c_2 \sin(\sqrt{\lambda})$$

$$y' = -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}c_2 \sin(\sqrt{\lambda})$$

$$-\sqrt{\lambda}c_1 \cos(0) + \sqrt{\lambda}c_2 \sin(0) = -\sqrt{\lambda}c_1 \cos(\sqrt{\lambda}) + \sqrt{\lambda}c_2 \sin(\sqrt{\lambda})$$

$$-\sqrt{\lambda}c_1 = -\sqrt{\lambda}c_1 \cos(\sqrt{\lambda}) + \sqrt{\lambda}c_2 \sin(\sqrt{\lambda})$$

$$-c_1 = -c_1 \cos(\sqrt{\lambda}) + c_2 \sin(\sqrt{\lambda})$$

$$c_1 = c_1 \cos(\sqrt{\lambda}) - c_2 \sin(\sqrt{\lambda})$$

$$c_1 - c_1 \cos(\sqrt{\lambda}) = -c_2 \sin(\sqrt{\lambda})$$

$$c_2 \sin(\sqrt{\lambda}) = -c_2 \sin(\sqrt{\lambda})$$

$$c_2 = 0$$

$$c_1 = c_1 \cos(\sqrt{\lambda})$$

$$1 = \cos(\sqrt{\lambda})$$

$$\lambda_n = (2n\pi)^2$$

$$y_n = c_1 \cos(2n\pi x) + c_2 \sin(2n\pi x)$$

Case $\lambda = 0$

$$r^2 = 0$$

$$r = 0$$

$$y = c_1 e^0 + c_2 x e^0$$

$$= c_1 + c_2 x$$

$$c_1 + 0 = c_1 + c_2$$

$$c_2 = 0$$

$$y' = c_2$$

$$y' = 0$$

Case $\lambda < 0$

$$r^2 - \lambda = 0$$

$$r = \pm\sqrt{\lambda}$$

$$y = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

$$c_1 e^0 + c_2 e^0 = c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}}$$

$$c_1 + c_2 = c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}}$$

$$c_1 = c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}} - c_2$$

$$y' = \sqrt{\lambda}c_1 e^{\sqrt{\lambda}x} - \sqrt{\lambda}c_2 e^{-\sqrt{\lambda}x}$$

$$\sqrt{\lambda}c_1 e^0 - \sqrt{\lambda}c_2 e^0 = \sqrt{\lambda}c_1 e^{\sqrt{\lambda}} - \sqrt{\lambda}c_2 e^{-\sqrt{\lambda}}$$

$$c_1 - c_2 = c_1 e^{\sqrt{\lambda}} - c_2 e^{-\sqrt{\lambda}}$$

$$c_1 = c_1 e^{\sqrt{\lambda}} - c_2 e^{-\sqrt{\lambda}} + c_2$$

$$c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}} - c_2 = c_1 e^{\sqrt{\lambda}} - c_2 e^{-\sqrt{\lambda}} + c_2$$

$$e^{-\sqrt{\lambda}} - 1 = -e^{-\sqrt{\lambda}} + 1$$

$$2e^{-\sqrt{\lambda}} = 2$$

$$e^{-\sqrt{\lambda}} = 1$$

$$\lambda = 0$$

Problem 4

Please complete Exercise 8.2.5 in the textbook. Begin your answer with the solution we that we derived in class:

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t}$$

Solve the BVP

$$\begin{aligned}
 u_t &= u_{xx} \quad 0 < x < 2 \quad t > 0 \\
 u_x(0, t) &= u(2, t) = 0 \quad t \geq 0 \\
 u(x, 0) &= \begin{cases} 1 & , 0 < x < 1 \\ 2 - x & , 1 < x < 2 \end{cases} \\
 &= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \\
 c_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\
 &= \int_0^1 \sin\left(\frac{n\pi x}{2}\right) dx + \int_1^2 (2 - x) \sin\left(\frac{n\pi x}{2}\right) dx \\
 &= \int_0^1 \sin\left(\frac{n\pi x}{2}\right) dx + \int_1^2 2 \sin\left(\frac{n\pi x}{2}\right) dx - \int_1^2 x \sin\left(\frac{n\pi x}{2}\right) dx \\
 &= \left[\frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right) \right]_0^1 + 2 \left[\frac{-2}{n\pi} \cos\left(\frac{n\pi x}{2}\right) \right]_1^2 - \frac{4 \cos(n\pi)}{n\pi} + \left[\frac{-4}{n^2 \pi^2} \sin\left(\frac{n\pi x}{2}\right) \right]_1^2 \\
 &= \frac{2}{n\pi} (1 - \cos(\frac{n\pi}{2})) + \frac{4}{n\pi} (\cos(\frac{n\pi}{2}) - \cos(n\pi)) - \frac{4}{n\pi} \cos(n\pi) + \frac{4}{n\pi} \sin(\frac{n\pi}{2}) \\
 &= \frac{2}{n\pi} (1 - \cos(\frac{n\pi}{2})) + \frac{4}{n\pi} (\cos(\frac{n\pi}{2}) - \cos(n\pi) - \cos(n\pi) + \sin(\frac{n\pi}{2})) \\
 &= \frac{2}{n\pi} (1 - \cos(\frac{n\pi}{2})) + \frac{4}{n\pi} (\cos(\frac{n\pi}{2}) - 2 \cos(n\pi) + \sin(\frac{n\pi}{2})) \\
 u(x, t) &= \sum_{n=1}^{\infty} \left(\frac{2}{n\pi} (1 - \cos(\frac{n\pi}{2})) + \frac{4}{n\pi} (\cos(\frac{n\pi}{2}) - 2 \cos(n\pi) + \sin(\frac{n\pi}{2})) \right) \sin\left(\frac{n\pi x}{L}\right) e^{-\beta(\frac{n\pi}{L})^2 t}
 \end{aligned}$$

Problem 5

Starting with separation of variables, find a formal solution to

$$\begin{aligned}
 u_t &= a^2 u_{xx} \quad 0 < x < L \quad t > 0 \\
 \frac{\partial u}{\partial x}(0, t) &= \frac{\partial u}{\partial x}(L, t) = 0 \quad t \geq 0 \\
 u(x, 0) &= f(x) \quad 0 < x < L \\
 u(x, t) &= X(x)T(t) \\
 XT' &= a^2 X''T \\
 \frac{X''}{X} &= \frac{T'}{a^2 T} = -\lambda
 \end{aligned}$$

$$X'' + \lambda X = 0$$

$$\lambda > 0$$

$$r^2 + \lambda = 0$$

$$r = 0 \pm \sqrt{\lambda}i$$

$$u = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

$$u_x = -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}x)$$

$$u_x(0, t) = 0 = 0 + \sqrt{\lambda}c_2 \cos(0)$$

$$c_2 = 0$$

$$u_x(L, t) = -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}L) = 0$$

$$\sqrt{\lambda}L = n\pi$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

$$\begin{aligned} u_n &= -\frac{n\pi}{L}c_1 \sin\left(\frac{n\pi x}{L}\right) \\ &= c \sin\left(\frac{n\pi x}{L}\right) \end{aligned}$$

$$\lambda = 0$$

$$r^2 = 0 \quad r = 0$$

$$u = c_1 e^0 + c_2 x e^0$$

$$u_x = c_2 = 0$$

$$\lambda > 0$$

$$r^2 - \lambda = 0$$

$$r = \pm\sqrt{\lambda}$$

$$u = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}$$

$$u_x = \sqrt{\lambda}c_1 e^{\sqrt{\lambda}x} - \sqrt{\lambda}c_2 e^{-\sqrt{\lambda}x}$$

$$u_x(0, t) = 0 = \sqrt{\lambda}c_1 - \sqrt{\lambda}c_2$$

$$c_1 = c_2$$

$$u_x(L, t) = 0 = e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}$$

$$\lambda = 0$$

$$T' + a^2\lambda T = 0$$

$$\frac{dT}{dt} = -\left(\frac{an\pi}{L}\right)^2 T$$

$$\int \frac{dT}{T} = -\left(\frac{an\pi}{L}\right)^2 \int dt$$

$$\ln |T| = -\left(\frac{an\pi}{L}\right)^2 t$$

$$T = e^{-\left(\frac{an\pi}{L}\right)^2 t}$$

$$u_n(x, t) = c_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{an\pi}{L}\right)^2 t\right)$$

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{an\pi}{L}\right)^2 t\right)$$

$$c_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx$$

Problem 6

Starting with the formal solution you found to the previous problem, complete Exercise 8.2.1 in the textbook.

Both faces of a bar of length L are insulated. The lateral surface of the bar is also insulated. The initial temperature in the bar is $\cos\left(\frac{3\pi x}{L}\right)$. Find the lateral temperature at any point x and time t for the bar. The BVP is

$$u_t(x, t) = a^2 u_{xx}(x, t) \quad 0 < x < L \quad t > 0$$

$$u_x(0, t) = u_x(L, t) = 0 \quad t \geq 0$$

$$u(x, 0) = \cos\left(\frac{3\pi x}{L}\right) \quad 0 < x < L$$

$$= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right)$$

$$c_n = \frac{2}{L} \int_0^L \cos\left(\frac{3\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{L} \int_0^L \frac{1}{2} \left[\sin\left(\frac{3\pi x}{L} + \frac{n\pi x}{L}\right) - \sin\left(\frac{3\pi x}{L} - \frac{n\pi x}{L}\right) \right] dx$$

$$= \frac{1}{2} \left(\int_0^L \sin\left(\frac{(3+n)\pi x}{L}\right) dx - \int_0^L \sin\left(\frac{(3-n)\pi x}{L}\right) dx \right)$$

$$= \frac{1}{2} \left(\frac{L}{(3+n)\pi} (1 - \cos((3+n)\pi)) - \frac{L}{(3-n)\pi} (1 - \cos((3-n)\pi)) \right)$$

$$u(x, t) = \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{L}{(3+n)\pi} (1 - \cos((3+n)\pi)) - \frac{L}{(3-n)\pi} (1 - \cos((3-n)\pi)) \right) \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{an\pi}{L}\right)^2 t\right)$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech