Boundary Value Problems: Homework 9

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Problem 1

Find all eigenvalues and eigenfunctions of the following boundary value problem:

$$y'' + \lambda y = 0, \quad y(0) = y'(2) = 0$$

$$\begin{aligned}
\text{Case } \lambda &> 0 \\
r^2 + \lambda &= 0 \\
r &= 0 \pm \sqrt{\lambda} i \\
y &= c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x) \\
y(0) &= 0 = c_1 \\
y' &= \sqrt{\lambda} c_2 \cos(\sqrt{\lambda} x) \\
y'(2) &= 0 = \sqrt{\lambda} c_2 \cos(2\sqrt{\lambda}) \\
0 &= \cos(2\sqrt{\lambda}) \\
2\sqrt{\lambda} &= (2n-1)\frac{\pi}{2} \\
\lambda_n &= \left(\frac{\pi(2n-1)}{4}\right)^2 \\
y_n &= c_2 \sin\left(\frac{\pi(2n-1)x}{4}\right)
\end{aligned}$$

$$\begin{aligned}
\text{Case } \lambda &= 0 \\
r^2 &= 0 \\
r &= 0 \\
y &= c_1 \mathrm{e}^0 + c_2 x \mathrm{e}^0 \\
&= c_1 + x c_2 \\
y(0) &= 0 = c_1 \\
y' &= c_1 \\
y'(2) &= 0 = c_2 \\
y &= 0
\end{aligned}$$

$$
\begin{aligned}
&\text{Case } \lambda < 0\\
r^2 - \lambda &= 0\\
r &= \pm\sqrt{\lambda}\\
y &= c_1 \mathrm{e}^{\sqrt{\lambda}x} + c_2 \mathrm{e}^{-\sqrt{\lambda}x}\\
y(0) &= 0 = c_1 + c_2\\
y' &= \sqrt{\lambda} c_1 \mathrm{e}^{\sqrt{\lambda}x} - \sqrt{\lambda} c_2 \mathrm{e}^{-\sqrt{\lambda}x}\\
y'(2) &= 0 = \sqrt{\lambda} c_1 \mathrm{e}^{2\sqrt{\lambda}x} + \sqrt{\lambda} c_1 \mathrm{e}^{-2\sqrt{\lambda}x}\\
&= \mathrm{e}^{2\sqrt{\lambda}x} + \mathrm{e}^{-2\sqrt{\lambda}x}\\
c_1 &= c_2 = 0
\end{aligned}
$$

Problem 2

Show that $y = c_1 \mathrm{e}^{\sqrt{-\lambda}x} + c_2 \mathrm{e}^{-\sqrt{-\lambda}x}$ can be rewritten as $y = d_1 \cosh(\sqrt{-\lambda}x) + d_2 \sinh(\sqrt{-\lambda}x)$ by choosing the values of constants d_1 and d_2 appropriately. Assume c_1 and c_2 are constants and that $\lambda < 0$.

$$
\begin{aligned}
y &= c_1 \mathrm{e}^{\sqrt{-\lambda}x} + c_2 \mathrm{e}^{-\sqrt{-\lambda}x}\\
Let: c_1 &= \frac{d_1 + d_2}{2} \quad c_2 = \frac{d_1 - d_2}{2}\\
&= (\frac{d_1 + d_2}{2}) \mathrm{e}^{\sqrt{-\lambda}x} + (\frac{d_1 - d_2}{2}) \mathrm{e}^{-\sqrt{-\lambda}x}\\
&= \frac{d_1}{2} \mathrm{e}^{\sqrt{-\lambda}x} + \frac{d_2}{2} \mathrm{e}^{\sqrt{-\lambda}x} + \frac{d_1}{2} \mathrm{e}^{-\sqrt{-\lambda}x} - \frac{d_2}{2} \mathrm{e}^{-\sqrt{-\lambda}x}\\
&= \frac{d_1}{2} \mathrm{e}^{\sqrt{-\lambda}x} + \frac{d_1}{2} \mathrm{e}^{-\sqrt{-\lambda}x} + \frac{d_2}{2} \mathrm{e}^{\sqrt{-\lambda}x} - \frac{d_2}{2} \mathrm{e}^{-\sqrt{-\lambda}x}\\
&= d_1 \left(\frac{\mathrm{e}^{\sqrt{-\lambda}x} + \mathrm{e}^{-\sqrt{-\lambda}x}}{2} \right) + d_2 \left(\frac{\mathrm{e}^{\sqrt{-\lambda}x} + \mathrm{e}^{-\sqrt{-\lambda}x}}{2} \right)\\
&= d_1 \cosh(\sqrt{-\lambda}x) + d_2 \sinh(\sqrt{-\lambda}x)
\end{aligned}
$$

Problem 3

Find all eigenvalues and eigenfunctions of the following boundary value problem:

$$
y'' + \lambda y = 0 \quad y(0) = y(1) \quad y'(0) = y'(1)
$$

$$
\begin{aligned}
\text{Case } \lambda &> 0 \\
r^2 + \lambda &= 0 \\
r &= 0 \pm \sqrt{\lambda} i \\
y &= c_1 \cos(\sqrt{\lambda} x) + c_2 \sin(\sqrt{\lambda} x) \\
c_1 \cos(0) + c_2 \sin(0) &= c_1 \cos(\sqrt{\lambda}) + c_2 \sin(\sqrt{\lambda}) \\
c_1 &= c_1 \cos(\sqrt{\lambda}) + c_2 \sin(\sqrt{\lambda}) \\
c_1 - c_1 \cos(\sqrt{\lambda}) &= c_2 \sin(\sqrt{\lambda}) \\
y' &= -\sqrt{\lambda} c_1 \sin(\sqrt{\lambda} x) + \sqrt{\lambda} c_2 \sin(\sqrt{\lambda}) \\
-\sqrt{\lambda} c_1 \cos(0) + \sqrt{\lambda} c_2 \sin(0) &= -\sqrt{\lambda} c_1 \cos(\sqrt{\lambda}) + \sqrt{\lambda} c_2 \sin(\sqrt{\lambda}) \\
-\sqrt{\lambda} c_1 &= -\sqrt{\lambda} c_1 \cos(\sqrt{\lambda}) + \sqrt{\lambda} c_2 \sin(\sqrt{\lambda}) \\
-c_1 &= -c_1 \cos(\sqrt{\lambda}) + c_2 \sin(\sqrt{\lambda}) \\
c_1 &= c_1 \cos(\sqrt{\lambda}) - c_2 \sin(\sqrt{\lambda}) \\
c_1 - c_1 \cos(\sqrt{\lambda}) &= -c_2 \sin(\sqrt{\lambda}) \\
c_2 \sin(\sqrt{\lambda}) &= -c_2 \sin(\sqrt{\lambda}) \\
c_2 &= 0 \\
c_1 &= c_1 \cos(\sqrt{\lambda}) \\
1 &= \cos(\sqrt{\lambda}) \\
\lambda_n &= (2n\pi)^2 \\
y_n &= c_1 \cos(2n\pi x) + c_2 \sin(2n\pi x)
\end{aligned}
$$

$$
\begin{aligned}
\text{Case } \lambda &= 0 \\
r^2 &= 0 \\
r &= 0 \\
y &= c_1 \mathrm{e}^0 + c_2 x \mathrm{e}^0 \\
&= c_1 + c_2 x \\
c_1 + 0 &= c_1 + c_2 \\
c_2 &= 0 \\
y' &= c_2 \\
y' &= 0
\end{aligned}
$$

$$\begin{aligned}
\text{Case } \lambda &< 0 \\
r^2 - \lambda &= 0 \\
r &= \pm\sqrt{\lambda} \\
y &= c_1 \mathrm{e}^{\sqrt{\lambda}x} + c_2 \mathrm{e}^{-\sqrt{\lambda}x} \\
c_1 \mathrm{e}^0 + c_2 \mathrm{e}^0 &= c_1 \mathrm{e}^{\sqrt{\lambda}} + c_2 \mathrm{e}^{-\sqrt{\lambda}} \\
c_1 + c_2 &= c_1 \mathrm{e}^{\sqrt{\lambda}} + c_2 \mathrm{e}^{-\sqrt{\lambda}} \\
c_1 &= c_1 \mathrm{e}^{\sqrt{\lambda}} + c_2 \mathrm{e}^{-\sqrt{\lambda}} - c_2 \\
y' &= \sqrt{\lambda} c_1 \mathrm{e}^{\sqrt{\lambda}x} - \sqrt{\lambda} c_2 \mathrm{e}^{-\sqrt{\lambda}x} \\
\sqrt{\lambda} c_1 \mathrm{e}^0 - \sqrt{\lambda} c_2 \mathrm{e}^0 &= \sqrt{\lambda} c_1 \mathrm{e}^{\sqrt{\lambda}} - \sqrt{\lambda} c_2 \mathrm{e}^{-\sqrt{\lambda}} \\
c_1 - c_2 &= c_1 \mathrm{e}^{\sqrt{\lambda}} - c_2 \mathrm{e}^{-\sqrt{\lambda}} \\
c_1 &= c_1 \mathrm{e}^{\sqrt{\lambda}} - c_2 \mathrm{e}^{-\sqrt{\lambda}} + c_2 \\
c_1 \mathrm{e}^{\sqrt{\lambda}} + c_2 \mathrm{e}^{-\sqrt{\lambda}} - c_2 &= c_1 \mathrm{e}^{\sqrt{\lambda}} - c_2 \mathrm{e}^{-\sqrt{\lambda}} + c_2 \\
\mathrm{e}^{-\sqrt{\lambda}} - 1 &= -\mathrm{e}^{-\sqrt{\lambda}} + 1 \\
2\mathrm{e}^{-\sqrt{\lambda}} &= 2 \\
\mathrm{e}^{-\sqrt{\lambda}} &= 1 \\
\lambda &= 0
\end{aligned}$$

Problem 4

Please complete Exercise 8.2.5 in the textbook. Begin your answer with the solution we that we derived in class:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L}) \mathrm{e}^{-\beta(\frac{n\pi}{L})^2 t}$$

Solve the BVP

$$
\begin{aligned}
u_t &= u_{xx} \quad 0 < x < 2 \quad t > 0 \\
u_x(0,t) &= u(2,t) = 0 \quad t \geq 0 \\
u(x,0) &= \begin{cases} 1 & ,0 < x < 1 \\ 2 - x & ,1 < x < 2 \end{cases} \\
&= \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L}) \\
c_n &= \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) \, \mathrm{d}x \\
&= \int_0^1 \sin(\frac{n\pi x}{2}) \, \mathrm{d}x + \int_1^2 (2-x) \sin(\frac{n\pi x}{2}) \, \mathrm{d}x \\
&= \int_0^1 \sin(\frac{n\pi x}{2}) \, \mathrm{d}x + \int_1^2 2\sin(\frac{n\pi x}{2}) \, \mathrm{d}x - \int_1^2 x \sin(\frac{n\pi x}{2}) \, \mathrm{d}x \\
&= \left[\frac{-2}{n\pi} \cos(\frac{n\pi x}{2})\right]_0^1 + 2\left[\frac{-2}{n\pi} \cos(\frac{n\pi x}{2})\right]_1^2 - \frac{4\cos(n\pi)}{n\pi} + \left[\frac{-4}{n^2\pi^2} \sin(\frac{n\pi x}{2})\right]_1^2 \\
&= \frac{2}{n\pi}(1 - \cos(\frac{n\pi}{2})) + \frac{4}{n\pi}(\cos(\frac{n\pi}{2}) - \cos(n\pi)) - \frac{4}{n\pi}\cos(n\pi) + \frac{4}{n\pi}\sin(\frac{n\pi}{2}) \\
&= \frac{2}{n\pi}(1 - \cos(\frac{n\pi}{2})) + \frac{4}{n\pi}(\cos(\frac{n\pi}{2}) - \cos(n\pi) - \cos(n\pi) + \sin(\frac{n\pi}{2})) \\
&= \frac{2}{n\pi}(1 - \cos(\frac{n\pi}{2})) + \frac{4}{n\pi}(\cos(\frac{n\pi}{2}) - 2\cos(n\pi) + \sin(\frac{n\pi}{2})) \\
u(x,t) &= \sum_{n=1}^{\infty} (\frac{2}{n\pi}(1 - \cos(\frac{n\pi}{2})) + \frac{4}{n\pi}(\cos(\frac{n\pi}{2}) - 2\cos(n\pi) + \sin(\frac{n\pi}{2}))) \sin(\frac{n\pi x}{L}) \mathrm{e}^{-\beta(\frac{n\pi}{L})^2 t}
\end{aligned}
$$

Problem 5

Starting with separation of variables, find a formal solution to

$$
\begin{aligned}
u_t &= a^2 u_{xx} \quad 0 < x < L \quad t > 0 \\
\frac{\partial u}{\partial x}(0,t) &= \frac{\partial u}{\partial x}(L,t) = 0 \quad t \geq 0 \\
u(x,0) &= f(x) \quad 0 < x < L \\
u(x,t) &= X(x)T(t) \\
XT' &= a^2 X'' T \\
\frac{X''}{X} &= \frac{T'}{a^2 T} = -\lambda
\end{aligned}
$$

$$
\begin{aligned}
X'' + \lambda X &= 0 \\
\lambda &> 0 \\
r^2 + \lambda &= 0 \\
r &= 0 \pm \sqrt{\lambda} i \\
u &= c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x) \\
u_x &= -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}x) + \sqrt{\lambda}c_2 \cos(\sqrt{\lambda}x) \\
u_x(0,t) &= 0 = 0 + \sqrt{\lambda}c_2 \cos(0) \\
c_2 &= 0 \\
u_x(L,t) &= -\sqrt{\lambda}c_1 \sin(\sqrt{\lambda}L) = 0 \\
\sqrt{\lambda}L &= n\pi \\
\lambda &= \left(\frac{n\pi}{L}\right)^2 \\
u_n &= -\frac{n\pi}{L} c_1 \sin(\frac{n\pi x}{L}) \\
&= c \sin(\frac{n\pi x}{L})
\end{aligned}
$$

$$
\begin{aligned}
\lambda &= 0 \\
r^2 &= 0 \quad r = 0 \\
u &= c_1 \mathrm{e}^0 + c_2 x \mathrm{e}^0 \\
u_x &= c_2 = 0
\end{aligned}
$$

$$
\begin{aligned}
\lambda &> 0 \\
r^2 - \lambda &= 0 \\
r &= \pm\sqrt{\lambda} \\
u &= c_1 \mathrm{e}^{\sqrt{\lambda}x} + c_2 \mathrm{e}^{-\sqrt{\lambda}x} \\
u_x &= \sqrt{\lambda}c_1 \mathrm{e}^{\sqrt{\lambda}x} - \sqrt{\lambda}c_2 \mathrm{e}^{-\sqrt{\lambda}x} \\
u_x(0,t) &= 0 = \sqrt{\lambda}c_1 - \sqrt{\lambda}c_2 \\
c_1 &= c_2 \\
u_x(L,t) &= 0 = \mathrm{e}^{\sqrt{\lambda}L} - \mathrm{e}^{-\sqrt{\lambda}L} \\
\lambda &= 0
\end{aligned}
$$

$$
\begin{aligned}
T' + a^2\lambda T &= 0 \\
\frac{\mathrm{d}T}{\mathrm{d}t} &= -(\frac{an\pi}{L})^2 T \\
\int \frac{\mathrm{d}T}{T} &= -(\frac{an\pi}{L})^2 \int \, \mathrm{d}t \\
\ln|T| &= -(\frac{an\pi}{L})^2 t \\
T &= \mathrm{e}^{-(\frac{an\pi}{L})^2 t} \\
u_n(x,t) &= c_n \sin(\frac{n\pi x}{L}) exp\left(-(\frac{an\pi}{L})^2 t\right)
\end{aligned}
$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L}) exp\left(-(\frac{an\pi}{L})^2 t\right)$$
$$c_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\ \mathrm{d}x$$

Problem 6

Starting with the formal solution you found to the previous problem, complete Exercise 8.2.1 in the textbook.

Both faces of a bar of length L are insulated. The lateral surface of the bar is also insulated. The initial temperature in the bar is $\cos(\frac{3\pi x}{L})$. Find the lateral temperature at any point x and time t for the bar. The BVP is

$$\begin{aligned}
u_t(x,t) &= a^2 u_{xx}(x,t) \quad 0 < x < L \quad t > 0 \\
u_x(0,t) &= u_x(L,t) = 0 \quad t \geq 0 \\
u(x,0) &= \cos(\frac{3\pi x}{L}) \quad 0 < x < L \\
&= \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L}) \\
c_n &= \frac{2}{L}\int_0^L \cos(\frac{3\pi x}{L})\sin(\frac{n\pi x}{L})\ \mathrm{d}x \\
&= \frac{2}{L}\int_0^L \frac{1}{2}\left[\sin(\frac{3\pi x}{L} + \frac{n\pi x}{L}) - \sin(\frac{3\pi x}{L} - \frac{n\pi x}{L})\right]\ \mathrm{d}x \\
&= \frac{1}{2}\left(\int_0^L \sin(\frac{(3+n)\pi x}{L})\ \mathrm{d}x - \int_0^L \sin(\frac{(3-n)\pi x}{L})\ \mathrm{d}x\right) \\
&= \frac{1}{2}\left(\frac{L}{(3+n)\pi}(1-\cos((3+n)\pi)) - \frac{L}{(3-n)\pi}(1-\cos((3-n)\pi))\right) \\
u(x,t) &= \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{L}{(3+n)\pi}(1-\cos((3+n)\pi)) - \frac{L}{(3-n)\pi}(1-\cos((3-n)\pi))\right)\sin(\frac{n\pi x}{L}) exp\left(-(\frac{an\pi}{L})^2 t\right)
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech