

Boundary Value Problems: Homework 8

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Problem 1

Use separation of variables to find, if possible, a family of nontrivial solutions, $u(x, y)$ for the PDE.

$$\begin{aligned}x \frac{\partial u}{\partial x} &= y \frac{\partial u}{\partial y} \\xX'(x)Y(y) &= yX(x)Y'(y) \\ \frac{xX'(x)}{X(x)} &= \frac{yY'(y)}{Y(y)} = \lambda \\xX'(x) &= \lambda X(x) \\x \frac{dX}{dx} &= \lambda X \\ \frac{dX}{X} &= \frac{\lambda}{x} dx \\ \ln |X| &= \lambda \ln |x| + c_1 \\ X &= e^{\lambda \ln |x| + c_1} \\ &= x^\lambda e^{c_1} \\ Y &= y^\lambda e^{c_2} \\ u(x, y) &= x^\lambda y^\lambda e^{c_3}\end{aligned}$$

Exercise 1a

Test the following PDEs for the method of separation of variables. If the method is successful, solve the PDE.

$$\begin{aligned}u_{xy} - u &= 0 \\X'(x)Y'(y) - X(x)Y(y) &= 0 \\X'(x)Y'(y) &= X(x)Y(y) \\ \frac{X'(x)}{X(x)} &= \frac{Y(y)}{Y'(y)} = \alpha^2 \\X'(x) - \alpha^2 X(x) &= 0 \\ X(x) &= Ae^{\alpha^2 x} \\ \alpha^2 Y'(y) - Y(y) &= 0 \\ Y(y) &= Be^{\frac{1}{\alpha^2} y} \\ u(x, y) &= X(x)Y(y) = Ce^{xy}\end{aligned}$$

Exercise 1c

Test the following PDEs for the method of separation of variables. If the method is successful, solve the PDE.

$$\begin{aligned}u_{xx} - u_{yy} - 2u_y &= 0 \\X''(x)Y(y) - X(x)Y''(y) - 2X(x)Y'(y) &= 0 \\X''Y &= X(Y'' + 2Y') \\ \frac{X''}{X} &= \frac{Y'' + 2Y'}{Y} = -\alpha^2 \\X'' + \alpha^2 X &= 0 \\X &= A \cos(\alpha x) + B \sin(\alpha x) \\Y'' + 2Y' + \alpha^2 Y &= 0 \\ \alpha^2 &= 0 \\Y(y) &= Ce^{-3y} + De^y \\u(x, y) &= X(x)Y(y) = (A \cos(\alpha x) + B \sin(\alpha x))(Ce^{-3y} + De^y)\end{aligned}$$

Exercise 2a

Find a solution for the boundary (or initial) value problem.

$$\begin{aligned}u_{tt} - u_{xx} &= 0 \quad u(L, 0) = u(0, t) = 0 \\X(x)T''(t) - X''(x)T(t) &= 0 \\XT'' &= X''T \\ \frac{X}{X''} &= \frac{T}{T''} = \alpha^2 \\ \alpha^2 X'' - X &= 0 \\ \alpha^2 = 0 \quad X &= 0 \\ \alpha^2 > 0 \quad X &= Ae^{\frac{x}{\alpha}} + Be^{-\frac{x}{\alpha}} \\ \alpha^2 < 0 \quad X &= A \cos\left(\frac{x}{\alpha}\right) + B \sin\left(\frac{x}{\alpha}\right) \\X(0) = 0 \quad X(L) = 0 &= B \sin\left(\frac{L}{\alpha}\right) \\ \frac{L}{\alpha} &= n\pi \quad \alpha = \frac{L}{n\pi} \\X &= C_1 \sin\left(\frac{n\pi x}{L}\right) \\Y &= C_2 \sin\left(\frac{m\pi x}{L}\right)\end{aligned}$$

Exercise 2b

Find a solution for the boundary (or initial) value problem.

$$\begin{aligned}u_{xx} - u_{yy} - u_y &= 0 & u_x(0, t) = u(L, 0) &= 0 \\X''Y - XY'' - XY' &= 0 \\X''Y &= XY'' + XY' \\ \frac{X''}{X} &= \frac{Y'' + Y'}{Y} = -\alpha^2 \\X'' + \alpha^2 X &= 0 \\X &= A \cos(\alpha x) + B \sin(\alpha x) \\X' &= A\alpha \sin(\alpha x) - B\alpha \cos(\alpha x) \\X(0) = 0 &= -B\alpha & B &= 0 \\X(L) = 0 &= A \cos(\alpha L) \\X(x) &= A \cos\left(\frac{(\frac{1}{2} + n)\pi x}{L}\right) \\Y'' + Y' + \alpha^2 Y &= 0 \\Y &= C + De^t(?)\end{aligned}$$

Exercise 2c

Find a solution for the boundary (or initial) value problem.

$$\begin{aligned}u_t = u_{xx} & \quad u_x(0, t) = 0 \\XT' = X''T \\ \frac{X}{X''} &= \frac{T}{T'} = -\alpha^2 \\\alpha^2 X'' + X &= 0 \\X &= A \cos\left(\frac{x}{\alpha}\right) + B \sin\left(\frac{x}{\alpha}\right) \\X' &= \frac{A}{\alpha} \sin\left(\frac{x}{\alpha}\right) - \frac{B}{\alpha} \cos\left(\frac{x}{\alpha}\right) \\X'(0) = 0 &= -\frac{B}{\alpha} & B &= 0 \\X &= A \cos\left(\frac{x}{\alpha}\right) \\\alpha^2 T' + T &= 0 \\\alpha^2 dT &= -T dt \\\alpha^2 T &= -\frac{T^2}{2} + C \\T &= -2\alpha^2 + C\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanagerd.tech