Boundary Value Problems: Homework 8

Alvin Lin

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Problem 1

Use separation of variables to find, if possible, a family of nontrivial solutions, $u(x, y)$ for the PDE.

$$\begin{aligned}
x\frac{\partial u}{\partial x} &= y\frac{\partial u}{\partial y} \\
xX'(x)Y(y) &= yX(x)Y'(y) \\
\frac{xX'(x)}{X(x)} &= \frac{yY'(y)}{Y(y)} = \lambda \\
xX'(x) &= \lambda X(x) \\
x\frac{\mathrm{d}X}{\mathrm{d}x} &= \lambda X \\
\frac{\mathrm{d}X}{X} &= \frac{\lambda}{x}\,\mathrm{d}x \\
\ln|X| &= \lambda\ln|x| + c_1 \\
X &= \mathrm{e}^{\lambda\ln|x|+c_1} \\
&= x^\lambda \mathrm{e}^{c_1} \\
Y &= y^\lambda \mathrm{e}^{c_2} \\
u(x, y) &= x^\lambda y^\lambda \mathrm{e}^{c_3}
\end{aligned}$$

Exercise 1a

Test the following PDEs for the method of separation of variables. If the method is successful, solve the PDE.

$$\begin{aligned}
u_{xy} - u &= 0 \\
X'(x)Y'(y) - X(x)Y(y) &= 0 \\
X'(x)Y'(y) &= X(x)Y(y) \\
\frac{X'(x)}{X(x)} &= \frac{Y(y)}{Y'(y)} = \alpha^2 \\
X'(x) - \alpha^2X(x) &= 0 \\
X(x) &= A\mathrm{e}^{\alpha^2x} \\
\alpha^2Y'(y) - Y(y) &= 0 \\
Y(y) &= B\mathrm{e}^{\frac{1}{\alpha^2}y} \\
u(x, y) &= X(x)Y(y) = C\mathrm{e}^{xy}
\end{aligned}$$

Exercise 1c

Test the following PDEs for the method of separation of variables. If the method is successful, solve the PDE.

$$
\begin{aligned}
u_{xx} - u_{yy} - 2u_y &= 0 \\
X''(x)Y(y) - X(x)Y''(y) - 2X(x)Y'(y) &= 0 \\
X''Y &= X(Y'' + 2Y') \\
\frac{X''}{X} &= \frac{Y'' + 2Y'}{Y} = -\alpha^2 \\
X'' + \alpha^2 X &= 0 \\
X &= A\cos(\alpha x) + B\sin(\alpha x) \\
Y'' + 2Y' + \alpha^2 Y &= 0 \\
\alpha^2 &= 0 \\
Y(y) &= C\mathrm{e}^{-3y} + D\mathrm{e}^{y} \\
u(x, y) &= X(x)Y(y) = (A\cos(\alpha x) + B\sin(\alpha x))(C\mathrm{e}^{-3y} + D\mathrm{e}^{y})
\end{aligned}
$$

Exercise 2a

Find a solution for the boundary (or initial) value problem.

$$
\begin{aligned}
u_{tt} - u_{xx} &= 0 \quad u(L, 0) = u(0, t) = 0 \\
X(x)T''(t) - X''(x)T(t) &= 0 \\
XT'' &= X''T \\
\frac{X}{X''} &= \frac{T}{T''} = \alpha^2 \\
\alpha^2 X'' - X &= 0 \\
\alpha^2 &= 0 \quad X = 0 \\
\alpha^2 &> 0 \quad X = A\mathrm{e}^{\frac{x}{\alpha}} + B\mathrm{e}^{\frac{x}{-\alpha}} \\
\alpha^2 &< 0 \quad X = A\cos(\frac{x}{\alpha}) + B\sin(\frac{x}{\alpha}) \\
X(0) &= 0 \quad X(L) = 0 = B\sin(\frac{L}{\alpha}) \\
\frac{L}{\alpha} &= n\pi \quad \alpha = \frac{L}{n\pi} \\
X &= C_1\sin(\frac{n\pi x}{L}) \\
Y &= C_2\sin(\frac{m\pi x}{L})
\end{aligned}
$$

Exercise 2b

Find a solution for the boundary (or initial) value problem.

$$
\begin{aligned}
u_{xx} - u_{yy} - u_y &= 0 \quad u_x(0,t) = u(L,0) = 0 \\
X''Y - XY'' - XY' &= 0 \\
X''Y &= XY'' + XY' \\
\frac{X''}{X} &= \frac{Y'' + Y'}{Y} = -\alpha^2 \\
X'' + \alpha^2 X &= 0 \\
X &= A\cos(\alpha x) + B\sin(\alpha x) \\
X' &= A\alpha\sin(\alpha x) - B\alpha\cos(\alpha x) \\
X(0) &= 0 = -B\alpha \quad B = 0 \\
X(L) &= 0 = A\cos(\alpha L) \\
X(x) &= A\cos\left(\frac{(\frac{1}{2}+n)\pi x}{L}\right) \\
Y'' + Y' + \alpha^2 Y &= 0 \\
Y &= C + De^t(?)
\end{aligned}
$$

Exercise 2c

Find a solution for the boundary (or initial) value problem.

$$
\begin{aligned}
u_t &= u_{xx} \quad u_x(0,t) = 0 \\
XT' &= X''T \\
\frac{X}{X''} &= \frac{T}{T'} = -\alpha^2 \\
\alpha^2 X'' + X &= 0 \\
X &= A\cos(\frac{x}{\alpha}) + B\sin(\frac{x}{\alpha}) \\
X' &= \frac{A}{\alpha}\sin(\frac{x}{\alpha}) - \frac{B}{\alpha}\cos(\frac{x}{\alpha}) \\
X'(0) &= 0 = -\frac{B}{\alpha} \quad B = 0 \\
X &= A\cos(\frac{x}{\alpha}) \\
\alpha^2 T' + T &= 0 \\
\alpha^2 \ \mathrm{d}T &= -T \ \mathrm{d}t \\
\alpha^2 T &= -\frac{T^2}{2} + C \\
T &= -2\alpha^2 + C
\end{aligned}
$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech