

Boundary Value Problems: Homework 6

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Problem 1

Write the Fourier cosine series and the Fourier sine series for $f(x) = \cos(x)$, $0 < x < \pi$.

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^\pi \cos(x) \, dx \\ &= \frac{2}{\pi} \left[\sin(x) \right]_0^\pi \\ &= \frac{2}{\pi} (\sin(\pi) - \sin(0)) \\ &= 0 \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi f(x) \cos\left(\frac{n\pi x}{\pi}\right) \, dx \\ &= \frac{2}{\pi} \int_0^\pi \cos(x) \cos(nx) \, dx \\ &= \frac{2}{\pi} \int_0^\pi \frac{1}{2} \left[\cos(x - nx) + \cos(x + nx) \right] \, dx \\ &= \frac{1}{\pi} \left[\frac{1}{1-n} \sin((1-n)x) + \frac{1}{1+n} \sin((1+n)x) \right]_0^\pi \\ &= 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi f(x) \sin\left(\frac{n\pi x}{\pi}\right) \, dx \\ &= \frac{2}{\pi} \int_0^\pi \cos(x) \sin(nx) \, dx \\ &= \frac{2}{\pi} \int_0^\pi \frac{1}{2} \left[\sin(x + nx) - \sin(x - nx) \right] \, dx \\ &= \frac{1}{\pi} \left[\frac{-1}{1+n} \cos(x + nx) + \frac{1}{1-n} \cos(x - nx) \right]_0^\pi \\ &= \frac{1}{\pi} \left[\frac{-\cos((n+1)\pi)}{1+n} + \frac{\cos((n-1)\pi)}{1-n} + \frac{\cos(0)}{1+n} - \frac{\cos(0)}{1-n} \right] \\ &= \frac{1 - \cos((1+n)\pi)}{(1+n)\pi} + \frac{\cos((1-n)\pi) - 1}{(1-n)\pi} \end{aligned}$$

$$FCS = 0$$

$$FSS = \sum_{n=1}^{\infty} \left(\frac{1 - \cos((1+n)\pi)}{(1+n)\pi} + \frac{\cos((1-n)\pi) - 1}{(1-n)\pi} \right) \sin(nx)$$

Problem 2

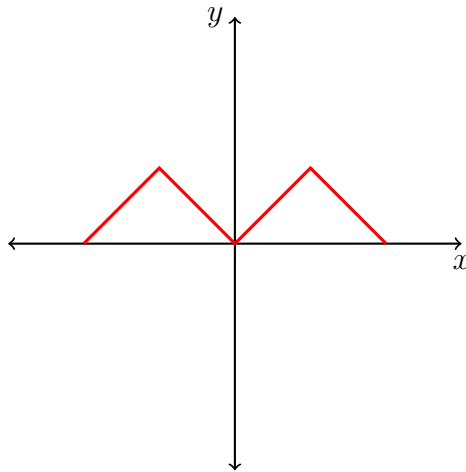
Find the Fourier cosine series for $f(x) = x, 0 \leq x \leq \pi$. If the series represents $f(x)$ on the given interval, show graphically the function represented by the series for all x .

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^\pi x \, dx \\ &= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^\pi \\ &= \frac{2}{\pi} \left[\frac{\pi^2}{2} - 0 \right] \\ &= \pi \end{aligned}$$

$$a_n = \frac{2}{\pi} \int_0^\pi x \cos\left(\frac{n\pi x}{\pi}\right) \, dx$$

$$\begin{aligned} u = x \quad du = dx \quad dv = \cos(nx) \quad v = \frac{-\sin(nx)}{n} \\ &= \frac{2}{n} \left[\left[\frac{x \sin(nx)}{n} \right]_0^\pi - \int_0^\pi \frac{1}{n} \sin(nx) \, dx \right] \\ &= \frac{2}{n} \left[\frac{\pi \sin(n\pi)}{n} - \left[\frac{-\cos(nx)}{n^2} \right]_0^\pi \right] \\ &= \frac{2}{n} \left[\frac{\pi \sin(n\pi)}{n} - \left[\frac{-\cos(n\pi)}{n^2} + \frac{\cos(0)}{n^2} \right] \right] \\ &= \frac{2\pi \sin(n\pi)}{n^2} - \frac{1 - \cos(n\pi)}{n^3} \end{aligned}$$

$$f(x) \sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \left(\frac{2\pi \sin(n\pi)}{n^2} - \frac{1 - \cos(n\pi)}{n^3} \right) \cos(nx)$$



Problem 3

If $f(x) = x^2, 0 < x < \pi$, find the Fourier sine series and draw a graph of the function with its periodic extension.

$$b_n = \frac{2}{\pi} \int_0^{\pi} x^2 \sin\left(\frac{n\pi x}{\pi}\right) dx$$

by integral table

$$\begin{aligned} &= \frac{2}{\pi} \left[\frac{(2 - n^2 x^2) \cos(nx) - 2nx \sin(nx)}{n^3} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{(2 - n^2 \pi^2) \cos(n\pi) - 2n\pi \sin(n\pi)}{n^3} - \frac{(2 - 0) \cos(0) - 0}{n^3} \right] \\ &= \frac{2}{\pi} \left[\frac{(2 - n^2 \pi^2) \cos(n\pi) - 2}{n^3} \right] \end{aligned}$$

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2}{\pi} \left[\frac{(2 - n^2 \pi^2) \cos(n\pi) - 2}{n^3} \right] \sin(nx)$$



Problem 4

Exercise 12: Find the double Fourier series if

$$f(x, y) = xy^2, -\pi < x < \pi, \pi < y < \pi$$

Since the function is odd in x and even in y , the coefficients a_{mn}, b_{mn}, d_{mn} are zero.

$$\begin{aligned}
c_{mn} &= \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} xy^2 \sin\left(\frac{m\pi x}{\pi}\right) \cos\left(\frac{n\pi y}{\pi}\right) dx dy \\
&= \frac{1}{\pi^2} \int_{-\pi}^{\pi} x \sin(mx) dx \int_{-\pi}^{\pi} y^2 \cos(ny) dy \\
&= \frac{1}{\pi^2} \left[\frac{\sin(mx) - mx \cos(mx)}{m^2} \right]_{-\pi}^{\pi} \left[\frac{(n^2 y^2 - 2) \sin(ny) + 2ny \cos(ny)}{n^3} \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi^2} \left[\frac{\sin(m\pi) - m\pi \cos(m\pi)}{m^2} - \frac{\sin(-m\pi) + m\pi \cos(-m\pi)}{m^2} \right] \\
&\quad \left[\frac{(n^2 \pi^2 - 2) \sin(n\pi) + 2n\pi \cos(n\pi)}{n^3} - \frac{(n^2 \pi^2 - 2) \sin(-n\pi) - 2n\pi \cos(-n\pi)}{n^3} \right] \\
&= \frac{1}{\pi^2} \left[\frac{-2m\pi \cos(m\pi)}{m^2} \right] \left[\frac{4n\pi \cos(n\pi)}{n^3} \right] \\
&= \frac{-8 \cos(m\pi) \cos(n\pi)}{mn^2} \\
f(x, y) &\sim \frac{1}{2} \sum_{m=1}^{\infty} \left[c_{m0} \sin\left(\frac{m\pi x}{K}\right) \right] + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[c_{mn} \cos\left(\frac{n\pi y}{L}\right) \right] \sin\left(\frac{m\pi x}{K}\right) \\
&= 0 + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{-8 \cos(m\pi) \cos(n\pi) \cos(ny)}{mn^2} \right] \sin(mx)
\end{aligned}$$

Exercise 13: Find the double Fourier series if

$$f(x, y) = x^2 y^2, \quad -\pi < x < \pi, \quad -\pi < y < \pi$$

Since the function is even in x and y , the coefficients b_{mn}, c_{mn}, d_{mn} are zero.

$$\begin{aligned}
a_{mn} &= \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} x^2 y^2 \cos(mx) \cos(ny) dx dy \\
&= \frac{1}{\pi^2} \int_{-\pi}^{\pi} x^2 \cos(mx) dx \int_{-\pi}^{\pi} y^2 \cos(ny) dy \\
&= \frac{1}{\pi^2} \left[\frac{(m^2 x^2 - 2) \sin(mx) + 2mx \cos(mx)}{m^3} \right]_{-\pi}^{\pi} \left[\frac{(n^2 y^2 - 2) \sin(ny) + 2ny \cos(ny)}{n^3} \right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi^2} \left[\frac{(m^2 \pi^2 - 2) \sin(m\pi) + 2m\pi \cos(m\pi)}{m^3} - \frac{(m^2 \pi^2 - 2) \sin(-m\pi) - 2m\pi \cos(m\pi)}{m^3} \right] \dots \\
&= \frac{1}{\pi^2} \left[\frac{4m\pi \cos(m\pi)}{m^3} \right] \left[\frac{4n\pi \cos(n\pi)}{n^3} \right] \\
&= \frac{16 \cos(m\pi) \cos(n\pi)}{(mn)^2} \\
f(x, y) &\sim \frac{a_{00}}{4} + \frac{1}{2} \sum_{n=1}^{\infty} a_{0n} \cos(ny) + \frac{1}{2} \sum_{m=1}^{\infty} a_{m0} \cos(mx) + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[a_{mn} \cos(ny) \right] \cos(mx) \\
&? = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[\frac{16 \cos(m\pi) \cos(n\pi) \cos(ny)}{(mn)^2} \right] \cos(mx)
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech