Boundary Value Problems: Homework 6

Alvin Lin

August 2018 - December 2018

Problem 1

Write the Fourier cosine series and the Fourier sine series for $f(x) = \cos(x), 0 < x < \pi$.

$$
\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^\pi \cos(x)\ \mathrm{d}x \\
&= \frac{2}{\pi}\Big[\sin(x)\Big]_0^\pi \\
&= \frac{2}{\pi}(\sin(\pi) - \sin(0)) \\
&= 0 \\
a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos(\frac{n\pi x}{\pi})\ \mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\pi \cos(x)\cos(nx)\ \mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\Big[\cos(x - nx) + \cos(x + nx)\Big]\ \mathrm{d}x \\
&= \frac{1}{\pi}\Big[\frac{1}{1-n}\sin((1-n)x) + \frac{1}{1+n}\sin((1+n)x)\Big]_0^\pi \\
&= 0 \\
b_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin(\frac{n\pi x}{\pi})\ \mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\pi \cos(x)\sin(nx)\ \mathrm{d}x \\
&= \frac{2}{\pi}\int_0^\pi \frac{1}{2}\Big[\sin(x + nx) - \sin(x - nx)\Big]\ \mathrm{d}x \\
&= \frac{1}{\pi}\Big[\frac{-1}{1+n}\cos(x + nx) + \frac{1}{1-n}\cos(x - nx)\Big]_0^\pi \\
&= \frac{1}{\pi}\Big[\frac{-\cos((n+1)\pi)}{1+n} + \frac{\cos((n-1)\pi)}{1-n} + \frac{\cos(0)}{1+n} - \frac{\cos(0)}{1-n}\Big] \\
&= \frac{1 - \cos((1+n)\pi)}{(1+n)\pi} + \frac{\cos((1-n)\pi) - 1}{(1-n)\pi} \\
FCS &= 0 \\
FSS &= \sum_{n=1}^{\infty}\left(\frac{1 - \cos((1+n)\pi)}{(1+n)\pi} + \frac{\cos((1-n)\pi) - 1}{(1-n)\pi}\right)\sin(nx)
\end{aligned}
$$

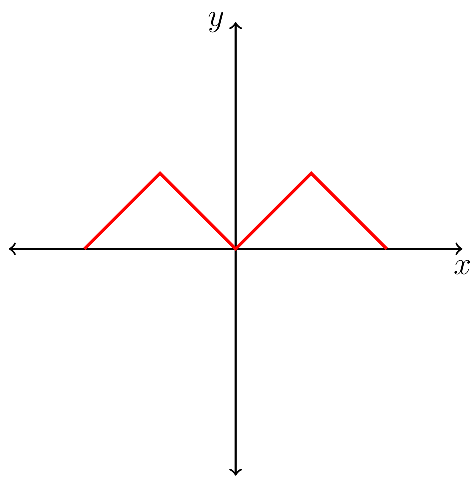
Problem 2

Find the Fourier cosine series for $f(x) = x, 0 \leq x \leq \pi$. If the series represents $f(x)$ on the given interval, show graphically the function represented by the series for all x.

$$\begin{aligned}
a_0 &= \frac{2}{\pi} \int_0^{\pi} x \ \mathrm{d}x \\
&= \frac{2}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi} \\
&= \frac{2}{\pi} \left[\frac{\pi^2}{2} - 0 \right] \\
&= \pi \\
a_n &= \frac{2}{\pi} \int_0^{\pi} x \cos(\frac{n\pi x}{\pi}) \ \mathrm{d}x
\end{aligned}$$

$$u = x \qquad \mathrm{d}u = \ \mathrm{d}x \qquad \mathrm{d}v = \cos(nx) \quad v = \frac{-\sin(nx)}{n}$$

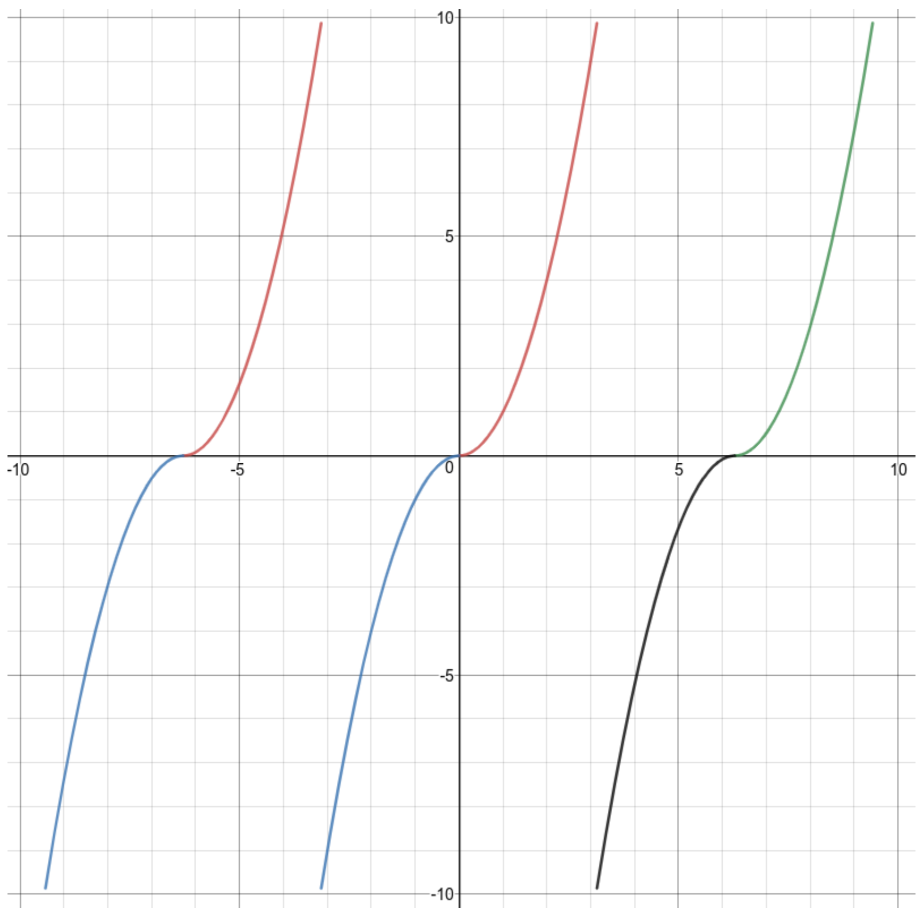
$$\begin{aligned}
&= \frac{2}{n} \left[\left[\frac{x \sin(nx)}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{1}{n} \sin(nx) \ \mathrm{d}x \right] \\
&= \frac{2}{n} \left[\frac{\pi \sin(n\pi)}{n} - \left[\frac{-\cos(nx)}{n^2} \right]_0^{\pi} \right] \\
&= \frac{2}{n} \left[\frac{\pi \sin(n\pi)}{n} - \left[\frac{-\cos(n\pi)}{n^2} + \frac{\cos(0)}{n^2} \right] \right] \\
&= \frac{2\pi \sin(n\pi)}{n^2} - \frac{1 - \cos(n\pi)}{n^3} \\
f(x) &\sim \frac{\pi}{2} + \sum_{n=1}^{\infty} \left(\frac{2\pi \sin(n\pi)}{n^2} - \frac{1 - \cos(n\pi)}{n^3} \right) \cos(nx)
\end{aligned}$$



Problem 3

If $f(x) = x^2, 0 < x < \pi$, find the Fourier sine series and draw a graph of the function with its periodic extension.

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x^2 \sin(\frac{n\pi x}{\pi})\ \mathrm{d}x \\ &\text{by integral table} \\ &= \frac{2}{\pi} \left[\frac{(2 - n^2x^2)\cos(nx) - 2nx\sin(nx)}{n^3} \right]_0^{\pi} \\ &= \frac{2}{\pi} \left[\frac{(2 - n^2\pi^2)\cos(n\pi) - 2n\pi\sin(n\pi)}{n^3} - \frac{(2-0)\cos(0) - 0}{n^3} \right] \\ &= \frac{2}{\pi} \left[\frac{(2 - n^2\pi^2)\cos(n\pi) - 2}{n^3} \right] \\ f(x) &\sim \sum_{n=1}^{\infty} \frac{2}{\pi} \left[\frac{(2 - n^2\pi^2)\cos(n\pi) - 2}{n^3} \right] \sin(nx) \end{aligned}$$



Problem 4

Exercise 12: Find the double Fourier series if

$$f(x, y) = xy^2, -\pi < x < \pi, \pi < y < \pi$$

Since the function is odd in x and even in y, the coefficients a_{mn}, b_{mn}, d_{mn} are zero.

$$\begin{aligned}
c_{mn} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} xy^2 \sin(\frac{m\pi x}{\pi})\cos(\frac{n\pi y}{\pi})\ \mathrm{d}x\ \mathrm{d}y \\
&= \frac{1}{\pi^2}\int_{-\pi}^{\pi} x\sin(mx)\ \mathrm{d}x \int_{-\pi}^{\pi} y^2\cos(ny)\ \mathrm{d}y \\
&= \frac{1}{\pi^2}\left[\frac{\sin(mx) - mx\cos(mx)}{m^2}\right]_{-\pi}^{\pi}\left[\frac{(n^2y^2-2)\sin(ny) + 2ny\cos(ny)}{n^3}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi^2}\left[\frac{\sin(m\pi) - m\pi\cos(m\pi)}{m^2} - \frac{\sin(-m\pi) + m\pi\cos(-m\pi)}{m^2}\right] \\
&\quad \left[\frac{(n^2\pi^2-2)\sin(n\pi) + 2n\pi\cos(n\pi)}{n^3} - \frac{(n^2\pi^2-2)\sin(-n\pi) - 2n\pi\cos(-n\pi)}{n^3}\right] \\
&= \frac{1}{\pi^2}\left[\frac{-2m\pi\cos(m\pi)}{m^2}\right]\left[\frac{4n\pi\cos(n\pi)}{n^3}\right] \\
&= \frac{-8\cos(m\pi)\cos(n\pi)}{mn^2}
\end{aligned}$$

$$\begin{aligned}
f(x,y) &\sim \frac{1}{2}\sum_{m=1}^{\infty}\left[c_{m0}\sin(\frac{m\pi x}{K})\right] + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[c_{mn}\cos(\frac{n\pi y}{L})\right]\sin(\frac{m\pi x}{K}) \\
&= 0 + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{-8\cos(m\pi)\cos(n\pi)\cos(ny)}{mn^2}\right]\sin(mx)
\end{aligned}$$

Exercise 13: Find the double Fourier series if

$$f(x,y) = x^2y^2, -\pi < x < \pi, -\pi < y < \pi$$

Since the function is even in x and y, the coefficients b_{mn}, c_{mn}, d_{mn} are zero.

$$\begin{aligned}
a_{mn} &= \frac{1}{\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} x^2y^2\cos(mx)\cos(ny)\ \mathrm{d}x\ \mathrm{d}y \\
&= \frac{1}{\pi^2}\int_{-\pi}^{\pi} x^2\cos(mx)\ \mathrm{d}x \int_{-\pi}^{\pi} y^2\cos(ny)\ \mathrm{d}y \\
&= \frac{1}{\pi^2}\left[\frac{(m^2x^2-2)\sin(mx) + 2mx\cos(mx)}{m^3}\right]_{-\pi}^{\pi}\left[\frac{(n^2y^2-2)\sin(ny) + 2ny\cos(ny)}{n^3}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi^2}\left[\frac{(m^2\pi^2-2)\sin(m\pi) + 2m\pi\cos(m\pi)}{m^3} - \frac{(m^2\pi^2-2)\sin(-m\pi) - 2m\pi\cos(m\pi)}{m^3}\right]\ldots \\
&= \frac{1}{\pi^2}\left[\frac{4m\pi\cos(m\pi)}{m^3}\right]\left[\frac{4n\pi\cos(n\pi)}{n^3}\right] \\
&= \frac{16\cos(m\pi)\cos(n\pi)}{(mn)^2}
\end{aligned}$$

$$\begin{aligned}
f(x,y) &\sim \frac{a_{00}}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_{0n}\cos(ny) + \frac{1}{2}\sum_{m=1}^{\infty} a_{m0}\cos(mx) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[a_{mn}\cos(ny)\right]\cos(mx) \\
? &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{16\cos(m\pi)\cos(n\pi)\cos(ny)}{(mn)^2}\right]\cos(mx)
\end{aligned}$$

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech