

Boundary Value Problems: Homework 5

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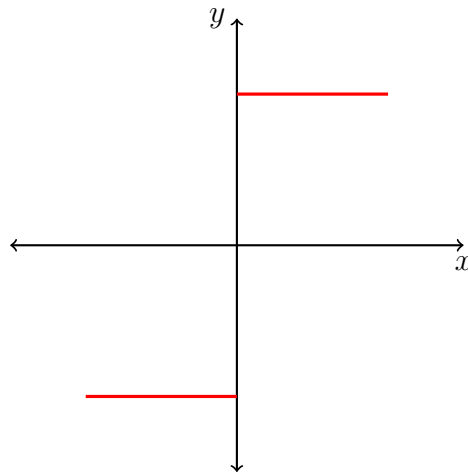
Problem 1

Exercises 3.2: problems 1, 2, 4, 5, 13

Exercise 1

Sketch the graph of f , determine the Fourier series corresponding to f , and indicate the convergence at the given points. It is assumed that the functions are periodic and one period is given.

$$f(x) = \begin{cases} -2 & , -2 < x < 0 \\ 2 & , 0 < x < 2 \end{cases}$$



$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-2}^2 f(x) \, dx \\ &= \frac{1}{2} \left[\int_{-2}^0 -2 \, dx + \int_0^2 2 \, dx \right] \\ &= 0 \\ a_n &= \frac{1}{2} \int_{-2}^2 f(x) \cos\left(\frac{n\pi x}{2}\right) \, dx \\ &= 0 \end{aligned}$$

$f(x)$ is characteristically odd, and since \cos is an even function, the resulting integral is odd.

$$\begin{aligned}
 b_n &= \frac{1}{2} \int_{-2}^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx \\
 &= \frac{1}{2} \left[\int_{-2}^0 -2 \sin\left(\frac{n\pi x}{2}\right) dx + \int_0^2 2 \sin\left(\frac{n\pi x}{2}\right) dx \right] \\
 &= \frac{1}{2} \left(\left[\frac{2}{n\pi} 2 \cos\left(\frac{n\pi x}{2}\right) \right]_{-2}^0 - \left[\frac{2}{n\pi} 2 \cos\left(\frac{n\pi x}{2}\right) \right]_0^2 \right) \\
 &= \frac{2}{n\pi} \cos(-n\pi) - \frac{2}{n\pi} \cos(0) - \frac{2}{n\pi} \cos(n\pi) + \frac{2}{n\pi} \cos(0) \\
 &= 0 \\
 f(x) &\sim 0
 \end{aligned}$$

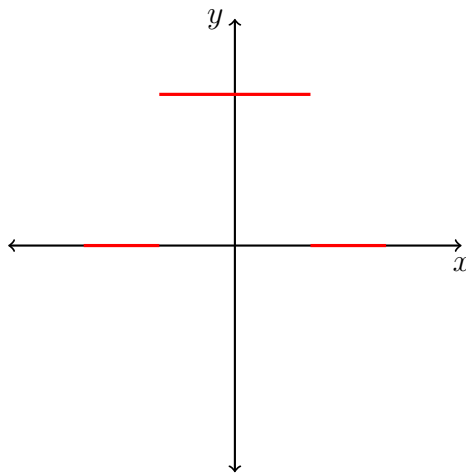
At the point $x = 0$,

$$\frac{f(0^-) + f(0^+)}{2} = \frac{-2 + 2}{2} = 0$$

Exercise 2

Sketch the graph of f , determine the Fourier series corresponding to f , and indicate the convergence at the given points. It is assumed that the functions are periodic and one period is given.

$$f(x) = \begin{cases} 0 & , -2 < x < -1 \\ 2 & , -1 < x < 1 \\ 0 & , 1 < x < 2 \end{cases}$$



$$\begin{aligned}
a_0 &= \frac{1}{2} \int_{-2}^2 f(x) \, dx \\
&= \frac{1}{2} \left(\int_{-2}^{-1} 0 \, dx + \int_{-1}^1 2 \, dx + \int_1^2 0 \, dx \right) \\
&= \frac{1}{2} \int_{-1}^1 2 \, dx \\
&= \frac{4}{2} \\
&= 2
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{1}{2} \int_{-2}^2 f(x) \cos\left(\frac{n\pi x}{2}\right) \, dx \\
&= \frac{1}{2} \int_{-1}^1 2 \cos\left(\frac{n\pi x}{2}\right) \, dx \\
&= \frac{1}{2} \left[\frac{2}{n\pi} (-2 \sin\left(\frac{n\pi x}{2}\right)) \right]_{-1}^1 \\
&= \frac{1}{2} \left[\frac{-4}{n\pi} \sin\left(\frac{n\pi}{2}\right) + \frac{4}{n\pi} \sin\left(\frac{-n\pi}{2}\right) \right] \\
&= \frac{1}{2} \left[\frac{-4}{n\pi} \sin\left(\frac{n\pi}{2}\right) - \frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right] \\
&= -\frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right)
\end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{2} \int_{-2}^2 f(x) \sin\left(\frac{n\pi x}{2}\right) \, dx \\
&= \frac{1}{2} \int_{-1}^1 2 \sin\left(\frac{n\pi x}{2}\right) \, dx \\
&= 0
\end{aligned}$$

$$\begin{aligned}
f(x) &\sim \frac{2}{2} + \sum_{n=1}^{\infty} -\frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi x}{2}\right) \\
&\sim 1 + \sum_{n=1}^{\infty} -\frac{4}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cos\left(\frac{n\pi x}{2}\right)
\end{aligned}$$

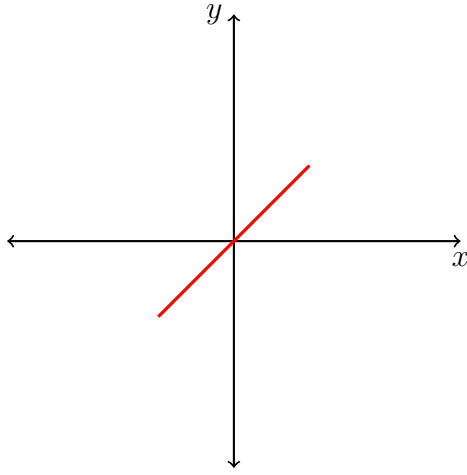
At the point $x = -1$,

$$\frac{f(-1^-) + f(-1^+)}{2} = \frac{0 + 2}{2} = 1$$

Exercise 3

Sketch the graph of f , determine the Fourier series corresponding to f , and indicate the convergence at the given points. It is assumed that the functions are periodic and one period is given.

$$f(x) = x, \quad -1 < x < 1$$



$$a_0 = \frac{1}{1} \int_{-1}^1 x \, dx$$

$$= 0$$

$$a_n = \int_{-1}^1 x \cos\left(\frac{n\pi x}{1}\right) \, dx$$

$$u = x \quad du = 1 \quad dv = \cos(n\pi x) \quad v = \frac{1}{n\pi} \sin(n\pi x)$$

$$= \left[\frac{1}{n\pi} x \sin(n\pi x) \right]_{-1}^1 - \int_{-1}^1 \frac{1}{n\pi} \sin(n\pi x) \, dx$$

$$= \left[\frac{1}{n\pi} \sin(n\pi) + \frac{1}{n\pi} \sin(-n\pi) \right] + \frac{1}{n\pi} \left[\frac{1}{n\pi} \cos(n\pi x) \right]_{-1}^1$$

$$= 0 - \frac{1}{n^2\pi^2} \left[\cos(n\pi) - \cos(-n\pi) \right]$$

$$= 0$$

$$b_n = \int_{-1}^1 x \sin\left(\frac{n\pi x}{1}\right) \, dx$$

$$u = x \quad du = 1 \quad dv = \sin(n\pi x) \quad v = -\frac{1}{n\pi} \cos(n\pi x)$$

$$= \left[-\frac{x}{n\pi} \cos(n\pi x) \right]_{-1}^1 - \int_{-1}^1 \frac{1}{n\pi} \cos(n\pi x) \, dx$$

$$= \left[-\frac{1}{n\pi} \cos(n\pi) - \frac{1}{n\pi} \cos(-n\pi) \right] - \frac{1}{n\pi} \left[\frac{1}{n\pi} \sin(n\pi x) \right]_{-1}^1$$

$$= -\frac{2 \cos(n\pi)}{n\pi} - \frac{1}{n^2\pi^2} \left[\sin(n\pi) - \sin(-n\pi) \right]$$

$$= -\frac{2 \cos(n\pi)}{n\pi} - \frac{2 \sin(n\pi)}{n^2\pi^2}$$

$$f(x) \sim 0 + \sum_{n=1}^{\infty} \left(-\frac{2 \cos(n\pi)}{n\pi} - \frac{2 \sin(n\pi)}{n^2\pi^2} \right) \sin(n\pi x)$$

$$\sim \sum_{n=1}^{\infty} -\frac{2 \cos(n\pi) \sin(n\pi x)}{n\pi} - \frac{2 \sin(n\pi) \sin(n\pi x)}{n^2\pi^2}$$

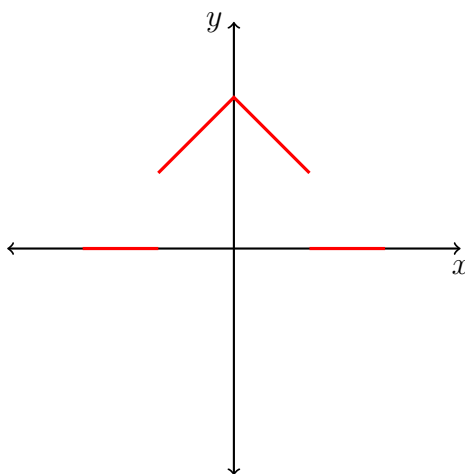
At the point $x = 1$,

$$\frac{f(1^-) + f(1^+)}{2} = \frac{1 + (-1)}{2} = 0$$

Exercise 4

Sketch the graph of f , determine the Fourier series corresponding to f , and indicate the convergence at the given points. It is assumed that the functions are periodic and one period is given.

$$f(x) = \begin{cases} 0 & , -2 < x < -1 \\ 2 + x & , -1 < x < 0 \\ 2 - x & , 0 < x < 1 \\ 0 & , 1 < x < 2 \end{cases}$$



$$\begin{aligned} a_0 &= \frac{1}{2} \int_{-2}^2 f(x) \, dx \\ &= \frac{1}{2} \left[\int_{-2}^{-1} 0 \, dx + \int_{-1}^0 (2+x) \, dx + \int_0^1 (2-x) \, dx + \int_1^2 0 \, dx \right] \\ &= \frac{1}{2} \left[0 + \frac{3}{2} + \frac{3}{2} + 0 \right] \\ &= \frac{3}{2} \\ a_n &= \frac{1}{2} \int_{-2}^2 f(x) \cos\left(\frac{n\pi x}{2}\right) \, dx \\ &= \frac{1}{2} \left[\int_{-1}^0 (2+x) \cos\left(\frac{n\pi x}{2}\right) + \int_0^1 (2-x) \cos\left(\frac{n\pi x}{2}\right) \right] \\ &= \frac{1}{2} \left[\frac{2(2+x)}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \Big|_{-1}^0 + \int_{-1}^0 \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \, dx + \right. \\ &\quad \left. \frac{2(2-x)}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \Big|_0^1 - \int_0^1 \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \, dx \right] \\ &= \frac{1}{2} \left[\frac{-2}{n\pi} \sin\left(\frac{-n\pi}{2}\right) + 1 + \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) - 0 \right] \\ &= \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + 1 \end{aligned}$$

$$\begin{aligned}
b_n &= \frac{1}{2} \int_{-2}^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx \\
&= 0
\end{aligned}$$

$f(x)$ is characteristically even and \sin is odd, so over a symmetric interval, the integral of their product is 0.

$$f(x) \sim \frac{2}{3} + \sum_{n=1}^{\infty} \left(\frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) + 1 \right) \cos\left(\frac{n\pi x}{2}\right)$$

At the point $x = 1$,

$$\frac{f(1^-) + f(1^+)}{2} = \frac{0 + 1}{2} = \frac{1}{2}$$

At the point $x = 2$,

$$\frac{f(2^-) + f(2^+)}{2} = \frac{0 + 0}{2} = 0$$

Exercise 5

Determine the Fourier series corresponding to f , and indicate the convergence at the given points. It is assumed that the functions are periodic and one period is given.

$$f(x) = e^{-x} \quad -1 < x < 1$$

$$\begin{aligned}
a_0 &= \frac{1}{1} \int_{-1}^1 f(x) dx \\
&= \int_{-1}^1 e^{-x} dx \\
&= \left[-e^{-x} \right]_{-1}^1 \\
&= -e^{-1} + e^1 \\
&= e - \frac{1}{e}
\end{aligned}$$

$$a_n = \int_{-1}^1 e^{-x} \cos\left(\frac{n\pi x}{1}\right) dx$$

$$\begin{aligned}
\left[\int e^{bx} \cos(ax) dx = \frac{1}{a^2 + b^2} e^{bx} (a \sin(ax) + b \cos(ax)) \right] \\
&= \left[\frac{1}{1 + n^2\pi^2} e^{-x} (n\pi \sin(n\pi x) - \cos(n\pi x)) \right]_{-1}^1 \\
&= \frac{e^{-1}(n\pi \sin(n\pi) - \cos(-n\pi))}{1 + n^2\pi^2} - \frac{e^1(n\pi \sin(-n\pi) - \cos(-n\pi))}{1 + n^2\pi^2} \\
&= \frac{-e^{-1} \cos(-n\pi) + e \cos(-n\pi)}{1 + n^2\pi^2} \\
&= \frac{\cos(n\pi)(e - \frac{1}{e})}{1 + n^2\pi^2}
\end{aligned}$$

$$\begin{aligned}
b_n &= \int_{-1}^1 e^{-x} \sin\left(\frac{n\pi x}{1}\right) dx \\
\left[\int e^{bx} \sin(ax) dx = \frac{1}{a^2 + b^2} e^{bx} (b \sin(ax) - a \cos(ax)) \right] \\
&= \left[\frac{1}{1 + n^2\pi^2} e^{-x} (-\sin(n\pi x) - n\pi \cos(n\pi x)) \right]_{-1}^1 \\
&= \frac{e^{-1}(-\sin(n\pi) - n\pi \cos(n\pi))}{1 + n^2\pi^2} - \frac{e(-\sin(-n\pi) - n\pi \cos(-n\pi))}{1 + n^2\pi^2} \\
&= \frac{-e^{-1}n\pi \cos(n\pi) + en\pi \cos(n\pi)}{1 + n^2\pi^2} \\
&= \frac{n\pi \cos(n\pi)(e - \frac{1}{e})}{1 + n^2\pi^2} \\
f(x) &\sim \frac{1}{e - \frac{1}{e}} + \sum_{n=1}^{\infty} \frac{(e - \frac{1}{e}) \cos(n\pi) \cos(n\pi x)}{1 + n^2\pi^2} + \frac{(e - \frac{1}{e})n\pi \cos(n\pi) \sin(n\pi x)}{1 + n^2\pi^2}
\end{aligned}$$

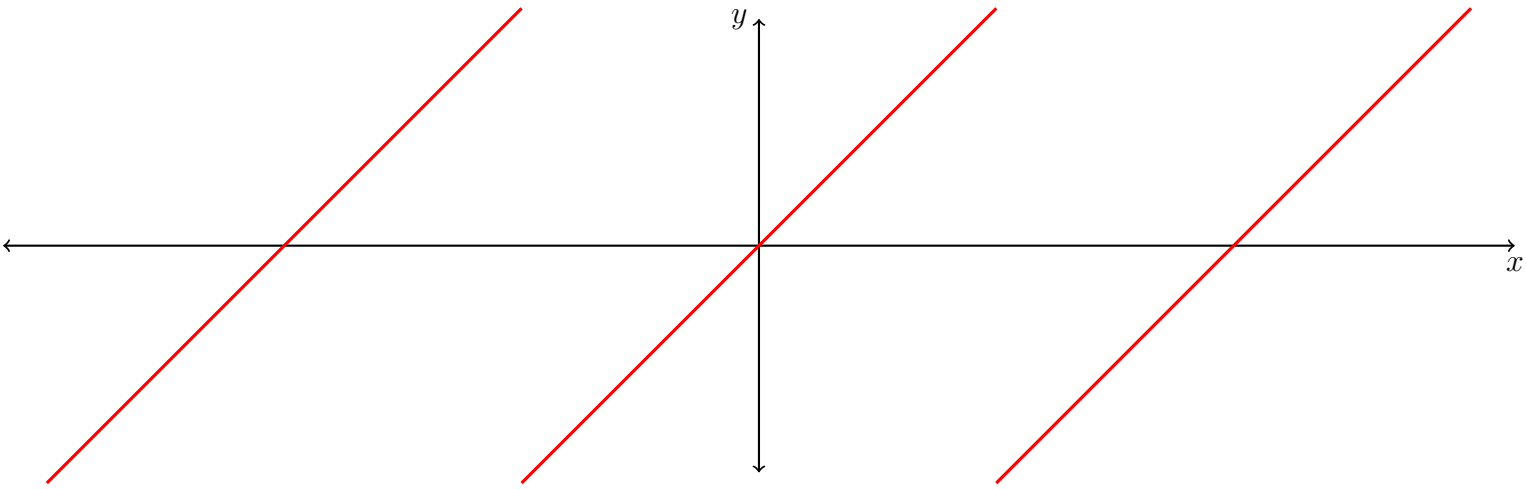
Find the convergence at $x = 1$ and $x = -1$.

$$\begin{aligned}
\frac{f(1^-) + f(1^+)}{2} &= \frac{e + e^{-1}}{2} \\
\frac{f(-1^-) + f(-1^+)}{2} &= \frac{e + e^{-1}}{2}
\end{aligned}$$

Exercise 13

If $f(x) = x$ for $-\pi < x < \pi$, find the Fourier series for the function. If the series represents $f(x)$ on the given interval show graphically the function represented by the series for all x .

$$\begin{aligned}
a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} x dx \\
&= 0 \\
a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \cos\left(\frac{n\pi x}{\pi}\right) dx \\
&= 0 \\
b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin\left(\frac{n\pi x}{\pi}\right) dx \\
u = x \quad du &= dx \quad dv = \sin(nx) \quad v = \frac{1}{n}(-\cos(nx)) \\
&= \left[\frac{-x \cos(nx)}{n} \right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{-\cos(nx)}{n} \\
&= \left[-\frac{\pi \cos(n\pi)}{n} - \frac{\pi \cos(n\pi)}{n} \right] - \left[\frac{-\sin(nx)}{n^2} \right]_{-\pi}^{\pi} \\
&= \frac{-2\pi \cos(n\pi)}{n} - [0 - 0] \\
&= \frac{-2\pi \cos(n\pi)}{n}
\end{aligned}$$



If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech