Boundary Value Problems: Homework 5

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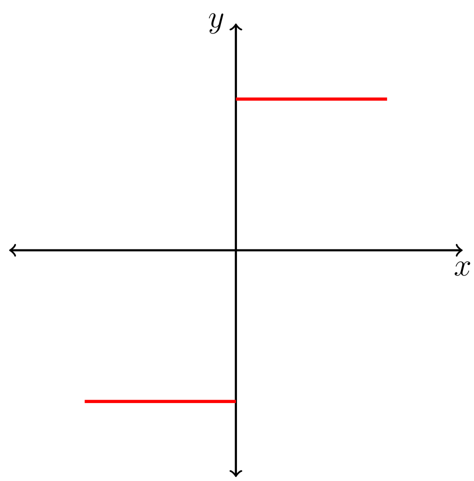
Problem 1

Exercises 3.2: problems 1, 2, 4, 5, 13

Exercise 1

Sketch the graph of f, determine the Fourier series corresponding to f, and indicate the convergence at the given points. It is assumed that the functions are periodic and one period is given.

$$f(x) = \begin{cases} -2 & ,-2 < x < 0 \\ 2 & ,0 < x < 2 \end{cases}$$



$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-2}^{2} f(x)\ \mathrm{d}x \\ &= \frac{1}{2}\Big[\int_{-2}^{0} -2\ \mathrm{d}x + \int_{0}^{2} 2\ \mathrm{d}x\Big] \\ &= 0 \\ a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos(\frac{n\pi x}{2})\ \mathrm{d}x \\ &= 0 \end{aligned}$$

$f(x)$ is characteristically odd, and since cos is an even function, the resulting integral is odd.

$$\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{n\pi x}{2})\ \mathrm{d}x \\
&= \frac{1}{2}\Big[\int_{-2}^{0} -2\sin(\frac{n\pi x}{2})\ \mathrm{d}x + \int_{0}^{2} 2\sin(\frac{n\pi x}{2})\ \mathrm{d}x\Big] \\
&= \frac{1}{2}\left(\Big[\frac{2}{n\pi}2\cos(\frac{n\pi x}{2})\Big]_{-2}^{0} - \Big[\frac{2}{n\pi}2\cos(\frac{n\pi x}{2})\Big]_{0}^{2}\right) \\
&= \frac{2}{n\pi}\cos(-n\pi) - \frac{2}{n\pi}\cos(0) - \frac{2}{n\pi}\cos(n\pi) + \frac{2}{n\pi}\cos(0) \\
&= 0 \\
f(x) &\sim 0
\end{aligned}$$

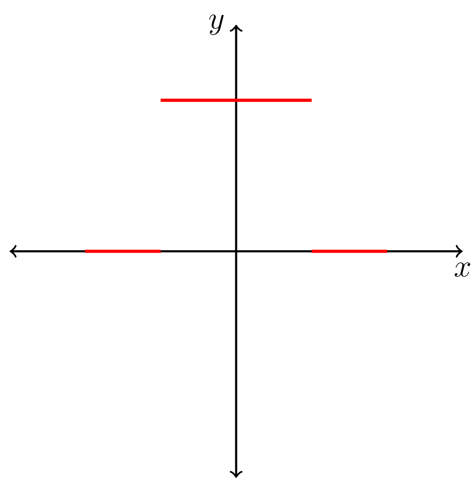
At the point $x = 0$,

$$\frac{f(0^-) + f(0^+)}{2} = \frac{-2+2}{2} = 0$$

Exercise 2

Sketch the graph of f, determine the Fourier series corresponding to f, and indicate the convergence at the given points. It is assumed that the functions are periodic and one period is given.

$$f(x) = \begin{cases} 0 & , -2 < x < -1 \\ 2 & , -1 < x < 1 \\ 0 & , 1 < x < 2 \end{cases}$$



$$
\begin{aligned}
a_0 &= \frac{1}{2}\int_{-2}^{2} f(x)\ \mathrm{d}x \\
&= \frac{1}{2}\left(\int_{-2}^{-1} 0\ \mathrm{d}x + \int_{-1}^{1} 2\ \mathrm{d}x + \int_{1}^{2} 0\ \mathrm{d}x\right) \\
&= \frac{1}{2}\int_{-1}^{1} 2\ \mathrm{d}x \\
&= \frac{4}{2} \\
&= 2 \\
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos(\frac{n\pi x}{2})\ \mathrm{d}x \\
&= \frac{1}{2}\int_{-1}^{1} 2\cos(\frac{n\pi x}{2})\ \mathrm{d}x \\
&= \frac{1}{2}\left[\frac{2}{n\pi}(-2\sin(\frac{n\pi x}{2}))\right]_{-1}^{1} \\
&= \frac{1}{2}\left[\frac{-4}{n\pi}\sin(\frac{n\pi}{2}) + \frac{4}{n\pi}\sin(\frac{-n\pi}{2})\right] \\
&= \frac{1}{2}\left[\frac{-4}{n\pi}\sin(\frac{n\pi}{2}) - \frac{4}{n\pi}\sin(\frac{n\pi}{2})\right] \\
&= -\frac{4}{n\pi}\sin(\frac{n\pi}{2}) \\
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{n\pi x}{2})\ \mathrm{d}x \\
&= \frac{1}{2}\int_{-1}^{1} 2\sin(\frac{n\pi x}{2})\ \mathrm{d}x \\
&= 0 \\
f(x) &\sim \frac{2}{2} + \sum_{n=1}^{\infty} -\frac{4}{n\pi}\sin(\frac{n\pi}{2})\cos(\frac{n\pi x}{2}) \\
&\sim 1 + \sum_{n=1}^{\infty} -\frac{4}{n\pi}\sin(\frac{n\pi}{2})\cos(\frac{n\pi x}{2})
\end{aligned}
$$

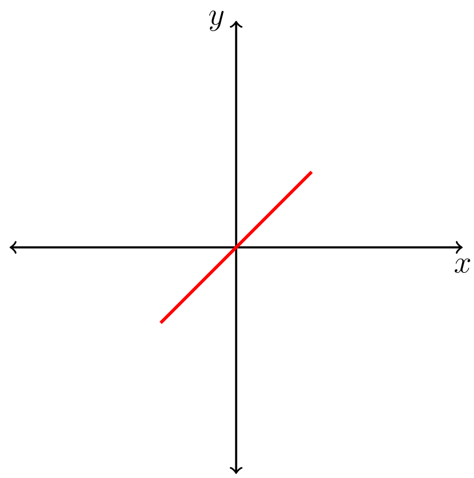
At the point $x = -1$,

$$\frac{f(-1^-) + f(-1^+)}{2} = \frac{0+2}{2} = 1$$

Exercise 3

Sketch the graph of f, determine the Fourier series corresponding to f, and indicate the convergence at the given points. It is assumed that the functions are periodic and one period is given.

$$f(x) = x, \quad -1 < x < 1$$



$$
\begin{aligned}
a_0 &= \frac{1}{1}\int_{-1}^{1} x \ \mathrm{d}x \\
&= 0 \\
a_n &= \int_{-1}^{1} x\cos(\frac{n\pi x}{1}) \ \mathrm{d}x \\
u = x \quad \mathrm{d}u &= 1 \quad \mathrm{d}v = \cos(n\pi x) \quad v = \frac{1}{n\pi}\sin(n\pi x) \\
&= \left[\frac{1}{n\pi}x\sin(n\pi x)\right]_{-1}^{1} - \int_{-1}^{1}\frac{1}{n\pi}\sin(n\pi x) \ \mathrm{d}x \\
&= \left[\frac{1}{n\pi}\sin(n\pi) + \frac{1}{n\pi}\sin(-n\pi)\right] + \frac{1}{n\pi}\left[\frac{1}{n\pi}\cos(n\pi x)\right]_{-1}^{1} \\
&= 0 - \frac{1}{n^2\pi^2}\left[\cos(n\pi) - \cos(-n\pi)\right] \\
&= 0 \\
b_n &= \int_{-1}^{1} x\sin(\frac{n\pi x}{1}) \ \mathrm{d}x \\
u = x \quad \mathrm{d}u &= 1 \quad \mathrm{d}v = \sin(n\pi x) \quad v = -\frac{1}{n\pi}\cos(n\pi x) \\
&= \left[-\frac{x}{n\pi}\cos(n\pi x)\right]_{-1}^{1} - \int_{-1}^{1}\frac{1}{n\pi}\cos(n\pi x) \ \mathrm{d}x \\
&= \left[-\frac{1}{n\pi}\cos(n\pi) - \frac{1}{n\pi}\cos(-n\pi)\right] - \frac{1}{n\pi}\left[\frac{1}{n\pi}\sin(n\pi x)\right]_{-1}^{1} \\
&= -\frac{2\cos(n\pi)}{n\pi} - \frac{1}{n^2\pi^2}\left[\sin(n\pi) - \sin(-n\pi)\right] \\
&= -\frac{2\cos(n\pi)}{n\pi} - \frac{2\sin(n\pi)}{n^2\pi^2} \\
f(x) &\sim 0 + \sum_{n=1}^{\infty}(-\frac{2\cos(n\pi)}{n\pi} - \frac{2\sin(n\pi)}{n^2\pi^2})\sin(n\pi x) \\
&\sim \sum_{n=1}^{\infty} -\frac{2\cos(n\pi)\sin(n\pi x)}{n\pi} - \frac{2\sin(n\pi)\sin(n\pi x)}{n^2\pi^2}
\end{aligned}
$$

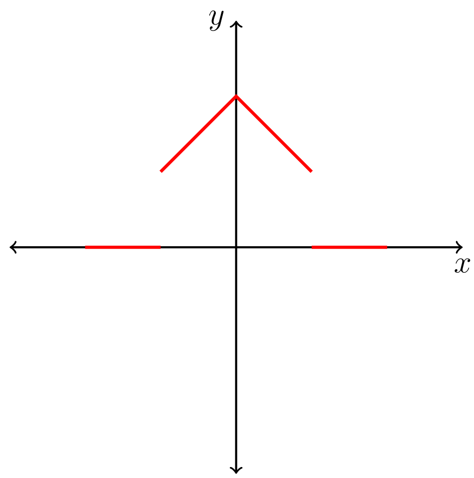
At the point $x = 1$,

$$\frac{f(1^-) + f(1^+)}{2} = \frac{1 + (-1)}{2} = 0$$

Exercise 4

Sketch the graph of f, determine the Fourier series corresponding to f, and indicate the convergence at the given points. It is assumed that the functions are periodic and one period is given.

$$f(x) = \begin{cases} 0 & , -2 < x < -1 \\ 2 + x & , -1 < x < 0 \\ 2 - x & , 0 < x < 1 \\ 0 & , 1 < x < 2 \end{cases}$$



$$\begin{aligned}
a_0 &= \frac{1}{2} \int_{-2}^{2} f(x) \, \mathrm{d}x \\
&= \frac{1}{2} \left[\int_{-2}^{1} 0 \, \mathrm{d}x + \int_{-1}^{0} 2 + x \, \mathrm{d}x + \int_{0}^{1} 2 - x \, \mathrm{d}x + \int_{1}^{2} 0 \, \mathrm{d}x \right] \\
&= \frac{1}{2} \left[0 + \frac{3}{2} + \frac{3}{2} + 0 \right] \\
&= \frac{3}{2} \\
a_n &= \frac{1}{2} \int_{-2}^{2} f(x) \cos(\frac{n\pi x}{2}) \, \mathrm{d}x \\
&= \frac{1}{2} \left[\int_{-1}^{0} (2 + x) \cos(\frac{n\pi x}{2}) + \int_{0}^{1} (2 - x) \cos(\frac{n\pi x}{2}) \right] \\
&= \frac{1}{2} \left[\frac{2(2 + x)}{n\pi} \sin(\frac{n\pi x}{2}) \Big|_{-1}^{0} + \int_{-1}^{0} \frac{2}{n\pi} \sin(\frac{n\pi x}{2}) \, \mathrm{d}x + \right. \\
&\qquad \left. \frac{2(2 - x)}{n\pi} \sin(\frac{n\pi x}{2}) \Big|_{0}^{1} - \int_{0}^{1} \frac{2}{n\pi} \sin(\frac{n\pi x}{2}) \, \mathrm{d}x \right] \\
&= \frac{1}{2} \left[\frac{-2}{n\pi} \sin(\frac{-n\pi}{2}) + 1 + \frac{2}{n\pi} \sin(\frac{n\pi}{2}) - 0 \right] \\
&= \frac{2}{n\pi} \sin(\frac{n\pi}{2}) + 1
\end{aligned}$$

$$
\begin{aligned}
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin(\frac{n\pi x}{2})\ \mathrm{d}x \\
&= 0
\end{aligned}
$$

$f(x)$ is characteristically even and sin is odd, so over a symmetric interval, the integral of their product is 0.

$$
f(x) \sim \frac{2}{3} + \sum_{n=1}^{\infty}(\frac{2}{n\pi}\sin(\frac{n\pi}{2}) + 1)\cos(\frac{n\pi x}{2})
$$

At the point $x = 1$,

$$
\frac{f(1^-) + f(1^+)}{2} = \frac{0+1}{2} = \frac{1}{2}
$$

At the point $x = 2$,

$$
\frac{f(2^-) + f(2^+)}{2} = \frac{0+0}{2} = 0
$$

Exercise 5

Determine the Fourier series corresponding to f, and indicate the convergence at the given points. It is assumed that the functions are periodic and one period is given.

$$
f(x) = \mathrm{e}^{-x} \qquad -1 < x < 1
$$

$$
\begin{aligned}
a_0 &= \frac{1}{1}\int_{-1}^{1} f(x)\ \mathrm{d}x \\
&= \int_{-1}^{1} \mathrm{e}^{-x}\ \mathrm{d}x \\
&= \Big[-\mathrm{e}^{-x}\Big]_{-1}^{1} \\
&= -\mathrm{e}^{-1} + \mathrm{e}^{1} \\
&= \mathrm{e} - \frac{1}{\mathrm{e}} \\
a_n &= \int_{-1}^{1} \mathrm{e}^{-x}\cos(\frac{n\pi x}{1})\ \mathrm{d}x \\
\Big[\int \mathrm{e}^{bx}\cos(ax)\ \mathrm{d}x &= \frac{1}{a^2+b^2}\mathrm{e}^{bx}(a\sin(ax) + b\cos(ax))\Big] \\
&= \Big[\frac{1}{1+n^2\pi^2}\mathrm{e}^{-x}(n\pi\sin(n\pi x) - \cos(n\pi x))\Big]_{-1}^{1} \\
&= \frac{\mathrm{e}^{-1}(n\pi\sin(n\pi) - \cos(-n\pi))}{1+n^2\pi^2} - \frac{\mathrm{e}^{1}(n\pi\sin(-n\pi) - \cos(-n\pi))}{1+n^2\pi^2} \\
&= \frac{-\mathrm{e}^{-1}\cos(-n\pi) + \mathrm{e}\cos(-n\pi)}{1+n^2\pi^2} \\
&= \frac{\cos(n\pi)(\mathrm{e} - \frac{1}{\mathrm{e}})}{1+n^2\pi^2}
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \int_{-1}^{1} \mathrm{e}^{-x} \sin(\frac{n\pi x}{1})\ \mathrm{d}x \\
\left[\int \mathrm{e}^{bx} \sin(ax)\ \mathrm{d}x\right. &= \left.\frac{1}{a^2+b^2}\mathrm{e}^{bx}(b\sin(ax) - a\cos(ax))\right] \\
&= \left[\frac{1}{1+n^2\pi^2}\mathrm{e}^{-x}(-\sin(n\pi x) - n\pi\cos(n\pi x))\right]_{-1}^{1} \\
&= \frac{\mathrm{e}^{-1}(-\sin(n\pi) - n\pi\cos(n\pi))}{1+n^2\pi^2} - \frac{\mathrm{e}(-\sin(-n\pi) - n\pi\cos(-n\pi))}{1+n^2\pi^2} \\
&= \frac{-e^{-1}n\pi\cos(n\pi) + \mathrm{e}n\pi\cos(n\pi)}{1+n^2\pi^2} \\
&= \frac{n\pi\cos(n\pi)(\mathrm{e} - \frac{1}{\mathrm{e}})}{1+n^2\pi^2} \\
f(x) &\sim \frac{1}{\mathrm{e}-\frac{1}{\mathrm{e}}} + \sum_{n=1}^{\infty} \frac{(\mathrm{e}-\frac{1}{\mathrm{e}})\cos(n\pi)\cos(n\pi x)}{1+n^2\pi^2} + \frac{(\mathrm{e}-\frac{1}{\mathrm{e}})n\pi\cos(n\pi)\sin(n\pi x)}{1+n^2\pi^2}
\end{aligned}
$$

Find the convergence at $x = 1$ and $x = -1$.

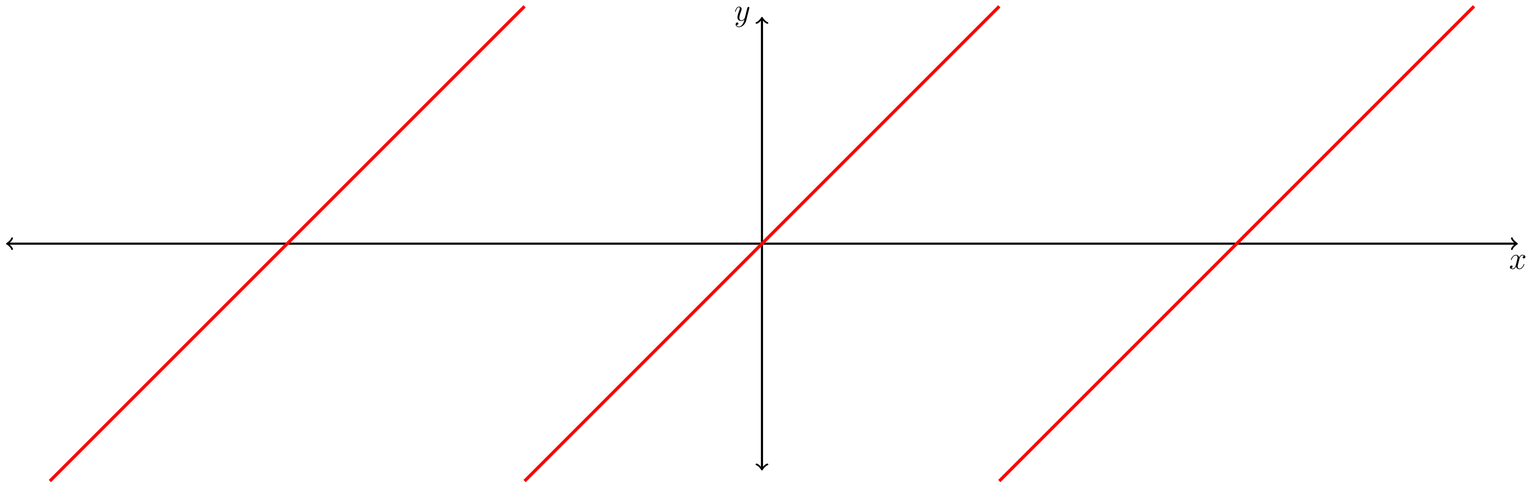
$$
\frac{f(1^-)+f(1^+)}{2} = \frac{\mathrm{e}+\mathrm{e}^{-1}}{2}
$$

$$
\frac{f(-1^-)+f(-1^+)}{2} = \frac{\mathrm{e}+\mathrm{e}^{-1}}{2}
$$

Exercise 13

If $f(x) = x$ for $-\pi < x < \pi$, find the Fourier series for the function. If the series represents $f(x)$ on the given interval show graphically the function represented by the series for all x.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\ \mathrm{d}x \\
&= 0 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(\frac{n\pi x}{\pi})\ \mathrm{d}x \\
&= 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(\frac{n\pi x}{\pi})\ \mathrm{d}x \\
u = x \quad \mathrm{d}u &= \ \mathrm{d}x \quad \mathrm{d}v = \sin(nx) \quad v = \frac{1}{n}(-\cos(nx)) \\
&= \left[\frac{-x\cos(nx)}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{-\cos(nx)}{n} \\
&= \left[-\frac{\pi\cos(n\pi)}{n} - \frac{\pi\cos(n\pi)}{n}\right] - \left[\frac{-\sin(nx)}{n^2}\right]_{-\pi}^{\pi} \\
&= \frac{-2\pi\cos(n\pi)}{n} - \left[0-0\right] \\
&= \frac{-2\pi\cos(n\pi)}{n}
\end{aligned}
$$



If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech