Boundary Value Problems: Homework 4

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Problem 1

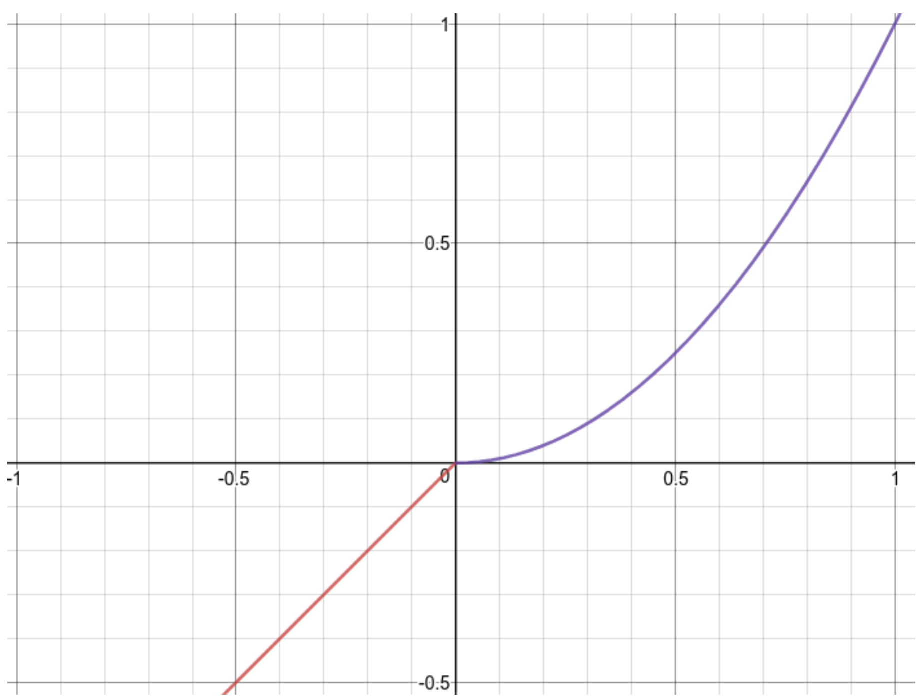
Exercises 3.1: problems 1, 2

Exercise 1

Graph the function

$$f(x) = \begin{cases} x & ,x < 0 \\ 1 & ,x = 0 \\ x^2 & ,x > 0 \end{cases}$$

Determine whether the function is PWC, continuous, PWS, and smooth.



This function is piecewise continuous and piecewise smooth.

Exercise 2

If $f(x) = |x|$, is $f(x)$ continuous at $x = 0$?

$$\lim_{x\to 0_+} f(x) = \lim_{x\to 0_-} f(x) = \lim_{x\to 0} f(x) = 0$$

$f(x)$ is continuous at 0.

$$f'(0_+) = 1$$
$$f'(0_-) = -1$$

$f(x)$ is not differentiable at 0 because the left and right hand derivatives differ on both sides.

Problem 2

Determine whether each of the following functions is piecewise smooth.

(a)

$$f(x) = \begin{cases} -x & , -\pi \leq x < 0 \\ 1 & , 0 \leq x \leq \pi \end{cases}$$

$$f'(x) = \begin{cases} -1 & , \pi \leq x < 0 \\ 0 & , 0 \leq x \leq \pi \end{cases}$$

This function is piecewise smooth.

(b)

$$f(x) = \begin{cases} 2 & , -5 \leq x < 0 \\ \frac{1}{x-1} & , 0 \leq x \leq 5 \end{cases}$$

$$f'(x) = \begin{cases} 0 & , -5 \leq x < 0 \\ \ln(x-1) & , 0 \leq x \leq 5 \end{cases}$$

This function is not piecewise smooth.

Problem 3

Suppose $C_n = (1 - \cos(n\pi))$, where $n \in \mathbb{N}$. Find expressions for C_{2n-1} and C_{2n} and simply the expressions and much as possible. The simplified expressions should just involve numbers, rather than cosines.

$$\begin{aligned} C_{2n-1} &= 1 - \cos((2n-1)\pi) \\ &= 1 - (-1) \\ &= 2 \quad \text{(2n-1 is an odd number)} \\ C_{2n} &= 1 - \cos(2n\pi) \\ &= 1 - (1) \\ &= 0 \quad \text{(2n is an even number)} \end{aligned}$$

Problem 4

Find the coefficients $a_0, a_n, b_n (n \in \mathbb{N})$, of the Fourier series for

$$f(x) = \begin{cases} 1 & , -2 \leq x < 0 \\ 0 & , 0 < x < 2 \end{cases}$$

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\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\ \mathrm{d}x \\
&= \frac{1}{2}(\int_{-2}^{0} 1\ \mathrm{d}x + \int_{0}^{2} 0\ \mathrm{d}x) \\
&= \frac{1}{2}(2+0) \\
&= 1 \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\ \mathrm{d}x \\
&= \frac{1}{2}\left(\int_{-2}^{0} 1\cos(\frac{n\pi x}{2})\ \mathrm{d}x + \int_{0}^{2} 0\ \mathrm{d}x\right) \\
&= \frac{1}{2}\frac{2}{n\pi}\left[\sin(\frac{n\pi x}{2})\right]_{-2}^{0} \\
&= \frac{1}{n\pi}(\sin(-n\pi) - \sin(0)) \\
&= 0 \\
b_n &= \frac{1}{2}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\ \mathrm{d}x \\
&= \frac{1}{2}\left(\int_{-2}^{0} 1\sin(\frac{n\pi x}{2})\ \mathrm{d}x + \int_{0}^{2} 0\ \mathrm{d}x\right) \\
&= \frac{1}{2}\frac{2}{n\pi}\left[-\cos(\frac{n\pi x}{2})\right]_{0}^{2} \\
&= \frac{1}{n\pi}(-\cos(n\pi) + \cos(0)) \\
&= \frac{1-\cos(n\pi)}{n\pi}
\end{aligned}
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If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech