Boundary Value Problems: Homework 3

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Problem 1

Exercises 2.1: problems 4-7

Exercise 4

Show that the set of functions $\{\sin(n\pi x), -1 < x < 1, n \in \mathbb{N}$ is orthogonal.

$$\begin{aligned}
n \in \mathbb{N}, m \in \mathbb{N}, n \neq m \\
(f_n, f_m) &= \int_{-1}^{1} \sin(m\pi x)\sin(n\pi x)\ \mathrm{d}x \\
&= \int_{-1}^{1} \frac{1}{2}\left(\cos(m\pi x - n\pi x) - \cos(m\pi x + n\pi x)\right)\ \mathrm{d}x \\
&= \frac{1}{2}\left[\frac{1}{m\pi - n\pi}\sin(m\pi x - n\pi x) + \frac{1}{m\pi + n\pi}\sin(m\pi x + n\pi x)\right]_{-1}^{1} \\
&= \frac{1}{2}\left[\frac{\sin(m\pi - n\pi)}{m\pi - n\pi} + \frac{\sin(m\pi + n\pi)}{m\pi + n\pi} - \frac{\sin(n\pi - m\pi)}{m\pi - n\pi} - \frac{\sin(-m\pi - n\pi)}{m\pi + n\pi}\right] \\
&= \frac{1}{2}(0 + 0 - 0 - 0) \\
&= 0
\end{aligned}$$

Because $m \in \mathbb{N}$ and $n \in \mathbb{N}$, all the sines evaluate to some $\sin(c\pi)$ where $c \in \mathbb{N}$. The sine of any multiple of π is 0.

$$\begin{aligned}
n \in \mathbb{N}, m \in \mathbb{N}, n = m \\
(f_n, f_m) &= \int_{-1}^{1} \frac{1}{2}\left(\cos(m\pi x - n\pi x) - \cos(m\pi x + n\pi x)\right)\ \mathrm{d}x \\
&= -\frac{1}{2}\int_{-1}^{1} \cos(2n\pi x)\ \mathrm{d}x \\
&= -\frac{1}{2}\left[\frac{\cos(2n\pi x)}{2n\pi}\right]_{-1}^{1} \\
&\neq 1
\end{aligned}$$

This set is not orthonormal.

Exercise 5

Find α so that $\{1, x, 1+\alpha x^2\}$ on $(-1,1)$ is orthogonal.

$$\begin{aligned}
0 &= (1, 1+\alpha x^2) \\
&= \int_{-1}^{1} 1+\alpha x^2 \, \mathrm{d}x \\
&= 2 + \alpha \left[\frac{x^3}{3}\right]_{-1}^{1} \\
&= 2 + \alpha \frac{2}{3} \\
-2 &= \frac{2\alpha}{3} \\
\alpha &= -3
\end{aligned}$$

Normalize the set.

$$\begin{aligned}
\|1\| &= \sqrt{\int_{-1}^{1} 1^2 \, \mathrm{d}x} \\
&= \sqrt{2} \\
\|x\| &= \sqrt{\int_{-1}^{1} x^2 \, \mathrm{d}x} \\
&= \sqrt{\frac{2}{3}} \\
\|1-3x^2\| &= \sqrt{\int_{-1}^{1} (1-3x^2)^2} \\
&= \sqrt{\int_{-1}^{1} 1 - 6x^2 + 9x^4 \, \mathrm{d}x} \\
&= \sqrt{\left[x - 2x^3 + \frac{9}{5}x^5\right]_{-1}^{1}} \\
&= \sqrt{1 - 2 + \frac{9}{5} + 1 - 2 + \frac{9}{5}} \\
&= \sqrt{\frac{18}{5} - 2} \\
&= \sqrt{\frac{8}{5}}
\end{aligned}$$

The normalized set is $\{\frac{1}{\sqrt{2}}, \frac{x\sqrt{3}}{\sqrt{2}}, \frac{(1-3x^2)\sqrt{5}}{2\sqrt{2}}\}$.

Exercise 6

Show that the set $\{1, \cos(\frac{n\pi x}{L}), \sin(\frac{m\pi x}{L})\}, n, m \in \mathbb{N}, -L < x < L$ is orthogonal but not orthonormal.

$$\begin{aligned}
(\cos(\frac{n\pi x}{L}), \sin(\frac{m\pi x}{L})) &= \int_{-L}^{L} \cos(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L}) \, \mathrm{d}x \\
&= \int_{-L}^{L} \frac{1}{2}(\sin(\frac{n\pi x + m\pi x}{L}) - \sin(\frac{n\pi x - m\pi x}{L})) \, \mathrm{d}x \\
&= 0 \\
(1, \sin(\frac{n\pi x}{L})) &= \int_{-L}^{L} \sin(\frac{n\pi x}{L}) \, \mathrm{d}x \\
&= 0
\end{aligned}$$

Because sin is an odd function over the symmetric integral $[-L, L]$.

$$\begin{aligned}
(1, \cos(\frac{n\pi x}{L})) &= \int_{-L}^{L} \cos(\frac{n\pi x}{L}) \, \mathrm{d}x \\
&= \left[\frac{L}{n\pi} \sin(\frac{n\pi x}{L})\right]_{-L}^{L} \\
&= \frac{L}{n\pi}(\sin(n\pi) - \sin(-n\pi)) \\
&= 0
\end{aligned}$$

$(f_m, f_n) = 0, m \neq n$, therefore the set is orthogonal.

$$\begin{aligned}
(\cos(\frac{n\pi x}{L}), \cos(\frac{n\pi x}{L})) &= \sqrt{\int_{-L}^{L} \cos^2(\frac{n\pi x}{L}) \, \mathrm{d}x} \\
&= \sqrt{\int_{-L}^{L} \frac{1 + \cos(\frac{2n\pi x}{L})}{2}} \\
&= \sqrt{L + \frac{1}{2}\left[\frac{L}{2n\pi} \sin(\frac{2n\pi x}{L})\right]_{-L}^{L}} \\
&\neq 1
\end{aligned}$$

Therefore, the set is not orthonormal.

Exercise 7

Is the set $\{\cos(\frac{n\pi x}{2})\}, n \in \mathbb{N}_0\}$, $0 < x < 2$, orthonormal? If it fails to be orthonormal, write the corresponding orthonormal set.

$$\begin{aligned}
& n \neq m \\
(\cos(\frac{n\pi x}{2}), \cos(\frac{m\pi x}{2})) &= \int_{0}^{2} \cos(\frac{n\pi x}{2}) \cos(\frac{m\pi x}{2}) \, \mathrm{d}x \\
&= \frac{1}{2}\int_{0}^{2} \cos(\frac{n\pi x - m\pi x}{2}) + \cos(\frac{n\pi x + m\pi x}{2}) \\
&= \left[\sin(n\pi x - m\pi x) + \sin(n\pi x + m\pi x)\right]_{0}^{2} \\
&= \sin(2n\pi - 2m\pi) + \sin(2n\pi + 2m\pi) - \sin(0) - \sin(0) \\
&= 0
\end{aligned}$$

The set is orthogonal.

$$
\begin{aligned}
n &= m \\
(\cos(\frac{n\pi x}{2}), \cos(\frac{n\pi x}{2})) &= \sqrt{\int_0^2 \cos^2(\frac{n\pi x}{2}) \, \mathrm{d}x} \\
&= \sqrt{\frac{1}{2}\int_0^2 1 + \cos(n\pi x) \, \mathrm{d}x} \\
&= \sqrt{\frac{1}{2}\left[x + \frac{\sin(n\pi x)}{n\pi}\right]_0^2} \\
&= \sqrt{\frac{1}{2}\left[2 + 0 - 0 - 0\right]} \\
&= \sqrt{1} \\
&= 1
\end{aligned}
$$

The set is also orthonormal.

Exercises 3.2: problems 6-8

Exercise 6

Prove that the sum of two odd functions is odd.

$$
\begin{aligned}
f(-x) &= -f(x) \\
\int_{-L}^{L} f(x) \, \mathrm{d}x &= 0 \\
g(-x) &= -g(x) \\
\int_{-L}^{L} g(x) \, \mathrm{d}x &= 0 \\
\int_{-L}^{L} f(x) + g(x) \, \mathrm{d}x &= \int_{-L}^{L} f(x) \, \mathrm{d}x + \int_{-L}^{L} g(x) \, \mathrm{d}x \\
&= 0 + 0 \\
&= 0
\end{aligned}
$$

Exercise 7

Show that if f is odd, then $|f|$ and f^2 are even functions.

$$
\begin{aligned}
f(-x) &= -f(x) \\
|f(-x)| &= |-f(x)| \\
f(-x) &= f(x)
\end{aligned}
$$

An odd function is symmetric about the origin. Taking the absolute value reflects the negative portion of the function across the x axis and because of the original symmetry, the function is now symmetric about the y axis.

$$\begin{aligned} f(-x) &= -f(x) \\ f(-x)^2 &= (-f(x))^2 \\ f(-x)^2 &= f(x)^2 \end{aligned}$$

Squaring an odd function takes the negative portion of the function and reflects it across the x axis due to the change in sign while increasing the whole function by a factor of itself. Because of the original symmetry, this function is now symmetric about the y axis.

Problem 2

Classify the following functions as even, odd, or neither. Show steps to justify your answers.

(a) $f(x) = (1 - x^2)^{-\frac{1}{2}}$

$$\begin{aligned} f(a) &= f(-a) \\ (1 - a^2)^{-\frac{1}{2}} &= (1 - a^2)^{-\frac{1}{2}} \end{aligned}$$

Even.

(b) $f(x) = \mathrm{e}^{-x}\cos(3x)$

$$\begin{aligned} f(a) &= f(-a) \\ \mathrm{e}^{-a}\cos(3a) &= \mathrm{e}^{a}\cos(-3a) \\ \mathrm{e}^{-a}\cos(3a) &\neq \mathrm{e}^{a}\cos(3a) \end{aligned}$$

Neither.

(c) $f(x) = \sinh(x)$

$$\begin{aligned} f(a) &= f(-a) \\ \sinh(a) &= \sinh(-a) \\ \sinh(a) &\neq -\sinh(a) \end{aligned}$$

Odd.

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech