

# Boundary Value Problems: Homework 2

Alvin Lin

August 2018 - December 2018

## Problem 1

Complete Exercises 1.2 problems 1, 2, 3, 5

### Exercise 1

Determine a general solution for the equation  $y'' + 5y' + 6y = 0$ .

$$\begin{aligned}y'' + 5y' + 6y &= 0 \\y &= e^{rt} \\r^2 + 5r + 6 &= (r + 2)(r + 3) = 0 \\r_1 &= -2 \quad r_2 = -3 \\y &= c_1 e^{-2t} + c_2 e^{-3t}\end{aligned}$$

### Exercise 2

Find a general solution for the equation  $y'' - 4y' + 4y = 0$ .

$$\begin{aligned}y'' - 4y' + 4y &= 0 \\r^2 - 4r + 4 &= 0 \\r_{1,2} &= -2 \\y &= e^{-2t} + te^{-2t}\end{aligned}$$

### Exercise 3

Solve the differential equation  $y'' + 2y' + 2y = 0$ .

$$\begin{aligned}y'' + 2y' + 2y &= 0 \\r^2 + 2r + 2 &= 0 \\r &= \frac{-2 + \sqrt{4 - (4)(1)(2)}}{2} \\&= -1 \pm i \quad \alpha = -1 \quad \beta = 1 \\y &= e^{-t} [c_1 \cos(t) + c_2 \sin(t)]\end{aligned}$$

### Exercise 5

Solve the boundary value problem  $y'' - y = 0$ ,  $y(0) = 0$ ,  $y'(\pi) = 1$ .

$$\begin{aligned}y'' - y &= 0 \\r^2 - 1 &= (r + 1)(r - 1) = 0 \\r_1 &= -1 \quad r_2 = 1 \\y &= c_1 e^{-t} + c_2 e^t \\y(0) = 0 &= c_1 + c_2 \\y &= c_1 e^{-t} - c_1 e^t \\y' &= -c_1 e^{-t} - c_1 e^t \\y'(\pi) = 1 &= -c_1 e^{-\pi} - c_1 e^{\pi} \\&= c_1(-e^{-\pi} - e^{\pi}) \\c_1 &= \frac{1}{-e^{-\pi} - e^{\pi}} \\y &= \frac{e^{-t}}{-e^{-\pi} - e^{\pi}} - \frac{e^t}{-e^{-\pi} - e^{\pi}}\end{aligned}$$

### Problem 2

Determine all solutions to

$$y'' + y = 0, y(0) = 0, y(2\pi) = 1$$

You may assume the domain is  $0 < x < 2\pi$ .

$$\begin{aligned}y'' + y &= 0 \\r^2 + 1 &= 0 \\r &= 0 \pm i \quad \alpha = 0 \quad \beta = 1 \\y &= e^0(c_1 \cos(t) + c_2 \sin(t)) \\&= c_1 \cos(t) + c_2 \sin(t) \\y(0) = 0 &= c_1 \cos(0) + c_2 \sin(0) \\c_1 &= 0 \\y(2\pi) = 1 &= c_2 \sin(2\pi) \\1 &= 0\end{aligned}$$

No solution.

### Problem 3

Determine all solutions to

$$y'' + 4y = 0, y'(0) = 0, y'(\frac{\pi}{2}) = 0$$

You may assume the domain is  $0 < x < \frac{\pi}{2}$ .

$$y'' + 4y = 0$$

$$r^2 + 4 = 0$$

$$r = 0 \pm 2i \quad \alpha = 0 \quad \beta = 2$$

$$y = e^{0t}(c_1 \cos(2t) + c_2 \sin(2t))$$

$$y' = -2c_1 \sin(2t) + 2c_2 \cos(2t)$$

$$y'(0) = 0 = -2c_1 \sin(0) + 2c_2 \cos(0)$$

$$c_2 = 0$$

$$y'(\frac{\pi}{2}) = 0 = -2c_1 \sin(\frac{\pi}{2})$$

$$c_1 = 0$$

$$y = 0$$

#### Problem 4

Classify the following PDEs as hyperbolic, parabolic, or elliptic.

(a)

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = -2 \frac{\partial^2 u}{\partial x^2} + 0.5 \frac{\partial u}{\partial x \partial t}$$

$$B = 2 \quad A = 1 \quad C = -0.5$$

$$\begin{aligned} B^2 - 4AC &= 4 - 4(1)(-0.5) \\ &= 6 \end{aligned}$$

hyperbolic

(b)

$$3u_{xx} + 2u_{xy} + u_y = 7u_{yy}$$

$$B = 2 \quad A = 3 \quad C = -7$$

$$\begin{aligned} B^2 - 4AC &= 4 - 4(3)(-7) \\ &= 88 \end{aligned}$$

hyperbolic

(c)

$$u_{yy} = u_x + u$$

$$B = 0 \quad A = 0 \quad C = 1$$

$$\begin{aligned} B^2 - 4AC &= 0 - 4(0)(1) \\ &= 0 \end{aligned}$$

parabolic

(d)

$$-3u_{yy} + 2u_{xy} = -u_{xx}$$

$$B = 2 \quad A = -3 \quad C = 1$$

$$\begin{aligned} B^2 - 4AC &= 4 - 4(-3)(1) \\ &= 16 \end{aligned}$$

hyperbolic

## Problem 5

Find all values of  $c \in \mathbb{R}$  such that the vectors  $\vec{a} = \langle c, 0.5, c \rangle$  and  $\vec{b} = \langle -3, 4, c \rangle$  are orthogonal.

$$\begin{aligned}\vec{a} \cdot \vec{b} &= 0 \\ -3c + 2 + c^2 &= 0 \\ (c - 2)(c - 1) &= 0 \\ c_1 &= 2 \quad c_2 = 1\end{aligned}$$

## Problem 6

Use the dot product to find the angle (in radians) between the vectors  $\vec{a} = \langle 2, 4, 0 \rangle$  and  $\vec{b} = \langle -1, -1, 4 \rangle$ .

$$\begin{aligned}\vec{a} \cdot \vec{b} &= \|\vec{a}\| \|\vec{b}\| \cos \theta \\ -2 + (-4) + 0 &= \sqrt{4 + 16} \sqrt{18} \cos \theta \\ -6 &= (2\sqrt{5})(3\sqrt{2}) \cos \theta \\ \cos \theta &= -\frac{1}{\sqrt{10}} \\ &= -\frac{\sqrt{10}}{10} \\ \theta &\approx 1.892\end{aligned}$$

## Problem 7

Let  $S = \{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ , where  $\vec{x}_1 = \langle 1, -1, 0 \rangle$ ,  $\vec{x}_2 = \langle 1, 1, 0 \rangle$ ,  $\vec{x}_3 = \langle 0, 0, 1 \rangle$ .

(a) Is  $S$  an orthogonal set? You will need to compute several dot products to answer this question.

$$\begin{aligned}\vec{x}_1 \cdot \vec{x}_2 &= 1 + (-1) + 0 = 0 \\ \vec{x}_1 \cdot \vec{x}_3 &= 0 + 0 + 0 = 0 \\ \vec{x}_2 \cdot \vec{x}_3 &= 0 + 0 + 0 = 0\end{aligned}$$

All the vectors are mutually orthogonal so the set is an orthogonal set.

(b) Is  $S$  an orthonormal set? Remember to show calculation(s) that support your answer.

$$\|\vec{x}_1\| = \sqrt{1 + 1} = \sqrt{2}$$

Since not all the vectors are unit vectors,  $S$  is not orthonormal.

If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)