Boundary Value Problems: Homework 2

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Problem 1

Complete Exercises 1.2 problems 1, 2, 3, 5

Exercise 1

Determine a general solution for the equation $y'' + 5y' + 6y = 0$.

$$\begin{aligned}
y'' + 5y' + 6y &= 0 \\
y &= \mathrm{e}^{rt} \\
r^2 + 5r + 6 &= (r+2)(r+3) = 0 \\
r_1 &= -2 \quad r_2 = -3 \\
y &= c_1\mathrm{e}^{-2t} + c_2\mathrm{e}^{-3t}
\end{aligned}$$

Exercise 2

Find a general solution for the equation $y'' - 4y' + 4y = 0$.

$$\begin{aligned}
y'' - 4y' + 4y &= 0 \\
r^2 - 4r + 4 &= 0 \\
r_{1,2} &= -2 \\
y &= \mathrm{e}^{-2t} + t\mathrm{e}^{-2t}
\end{aligned}$$

Exercise 3

Solve the differential equation $y'' + 2y' + 2y = 0$.

$$\begin{aligned}
y'' + 2y' + 2y &= 0 \\
r^2 + 2r + 2 &= 0 \\
r &= \frac{-2 + \sqrt{4 - (4)(1)(2)}}{2} \\
&= -1 \pm i \quad \alpha = -1 \quad \beta = 1 \\
y &= \mathrm{e}^{-t}\left[c_1\cos(t) + c_2\sin(t)\right]
\end{aligned}$$

Exercise 5

Solve the boundary value problem $y'' - y = 0, y(0) = 0, y'(\pi) = 1$.

$$\begin{aligned}
y'' - y &= 0 \\
r^2 - 1 &= (r+1)(r-1) = 0 \\
r_1 &= -1 \quad r_2 = 1 \\
y &= c_1 \mathrm{e}^{-t} + c_2 \mathrm{e}^{t} \\
y(0) &= 0 = c_1 + c_2 \\
y &= c_1 \mathrm{e}^{-t} - c_1 \mathrm{e}^{t} \\
y' &= -c_1 \mathrm{e}^{-t} - c_1 \mathrm{e}^{t} \\
y'(\pi) &= 1 = -c_1 \mathrm{e}^{-\pi} - c_1 \mathrm{e}^{\pi} \\
&= c_1(-\mathrm{e}^{-\pi} - \mathrm{e}^{\pi}) \\
c_1 &= \frac{1}{-\mathrm{e}^{-\pi} - \mathrm{e}^{\pi}} \\
y &= \frac{\mathrm{e}^{-t}}{-\mathrm{e}^{-\pi} - \mathrm{e}^{\pi}} - \frac{\mathrm{e}^{t}}{-\mathrm{e}^{-\pi} - \mathrm{e}^{\pi}}
\end{aligned}$$

Problem 2

Determine all solutions to

$$y'' + y = 0, y(0) = 0, y(2\pi) = 1$$

You may assume the domain is $0 < x < 2\pi$.

$$\begin{aligned}
y'' + y &= 0 \\
r^2 + 1 &= 0 \\
r &= 0 \pm i \quad \alpha = 0 \quad \beta = 1 \\
y &= \mathrm{e}^{0}(c_1 \cos(t) + c_2 \sin(t)) \\
&= c_1 \cos(t) + c_2 \sin(t) \\
y(0) &= 0 = c_1 \cos(0) + c_2 \sin(0) \\
c_1 &= 0 \\
y(2\pi) &= 1 = c_2 \sin(2\pi) \\
1 &= 0
\end{aligned}$$

No solution.

Problem 3

Determine all solutions to

$$y'' + 4y = 0, y'(0) = 0, y'(\frac{\pi}{2}) = 0$$

You may assume the domain is $0 < x < \frac{\pi}{2}$.

$$\begin{aligned}
y'' + 4y &= 0 \\
r^2 + 4 &= 0 \\
r &= 0 \pm 2i \quad \alpha = 0 \quad \beta = 2 \\
y &= \mathrm{e}^{0t}(c_1 \cos(2t) + c_2 \sin(2t)) \\
y' &= -2c_1 \sin(2t) + 2c_2 \cos(2t) \\
y'(0) &= 0 = -2c_1 \sin(0) + 2c_2 \cos(0) \\
c_2 &= 0 \\
y'(\frac{\pi}{2}) &= 0 = -2c_1 \sin(\frac{\pi}{2}) \\
c_1 &= 0 \\
y &= 0
\end{aligned}$$

Problem 4

Classify the following PDEs as hyperbolic, parabolic, or elliptic.

(a)

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial t^2} = -2\frac{\partial^2 u}{\partial x^2} + 0.5\frac{\partial^u}{\partial x \partial t}$$

$$\begin{aligned}
B = 2 \quad A = 1 \quad C &= -0.5 \\
B^2 - 4AC &= 4 - 4(1)(-0.5) \\
&= 6
\end{aligned}$$

hyperbolic

(b)

$$3u_{xx} + 2u_{xy} + u_y = 7u_{yy}$$

$$\begin{aligned}
B = 2 \quad A = 3 \quad C &= -7 \\
B^2 - 4AC &= 4 - 4(3)(-7) \\
&= 88
\end{aligned}$$

hyperbolic

(c)

$$u_{yy} = u_x + u$$

$$\begin{aligned}
B = 0 \quad A = 0 \quad C &= 1 \\
B^2 - 4AC &= 0 - 4(0)(1) \\
&= 0
\end{aligned}$$

parabolic

(d)

$$-3u_{yy} + 2u_{xy} = -u_{xx}$$

$$\begin{aligned}
B = 2 \quad A = -3 \quad C &= 1 \\
B^2 - 4AC &= 4 - 4(-3)(1) \\
&= 16
\end{aligned}$$

hyperbolic

Problem 5

Find all values of $c \in \mathbb{R}$ such that the vectors $\vec{a} = \langle c, 0.5, c\rangle$ and $\vec{b} = \langle -3, 4, c\rangle$ are orthogonal.

$$\begin{aligned} \vec{a} \cdot \vec{b} &= 0 \\ -3c + 2 + c^2 &= 0 \\ (c-2)(c-1) &= 0 \\ c_1 &= 2 \quad c_2 = 1 \end{aligned}$$

Problem 6

Use the dot product to find the angle (in radians) between the vectors $\vec{a} = \langle 2, 4, 0\rangle$ and $\vec{b} = \langle -1, -1, 4\rangle$.

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \|\vec{a}\|\|\vec{b}\| \cos\theta \\ -2 + (-4) + 0 &= \sqrt{4+16}\sqrt{18}\cos\theta \\ -6 &= (2\sqrt{5})(3\sqrt{2})\cos\theta \\ \cos\theta &= -\frac{1}{\sqrt{10}} \\ &= -\frac{\sqrt{10}}{10} \\ \theta &\approx 1.892 \end{aligned}$$

Problem 7

Let $S = \{\vec{x_1}, \vec{x_2}, \vec{x_3}\}$, where $\vec{x_1} = \langle 1, -1, 0\rangle, \vec{x_2} = \langle 1, 1, 0\rangle, \vec{x_3} = \langle 0, 0, 1\rangle$.

(a) Is S an orthogonal set? You will need to compute several dot products to answer this question.

$$\begin{aligned} \vec{x_1} \cdot \vec{x_2} &= 1 + (-1) + 0 = 0 \\ \vec{x_1} \cdot \vec{x_3} &= 0 + 0 + 0 = 0 \\ \vec{x_2} \cdot \vec{x_3} &= 0 + 0 + 0 = 0 \end{aligned}$$

All the vectors are mutually orthogonal so the set is an orthogonal set.

(b) Is S an orthonormal set? Remember to show calculation(s) that support your answer.

$$\|\vec{x_1}\| = \sqrt{1+1} = \sqrt{2}$$

Since not all the vectors are unit vectors, S is not orthonormal.

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech