Boundary Value Problems

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Laplace's Equation

In one dimension, the heat equation looks like:

$$u_t = a^2 u_{xx}$$

We will see that the heat equation in two dimensions looks like:

$$u_t = a^2(u_{xx} + u_{yy})$$

We can reach a steady state by letting $t \to \infty$ and $u_t \to 0$. The steady state temperature distribution is given by Laplace's equations:

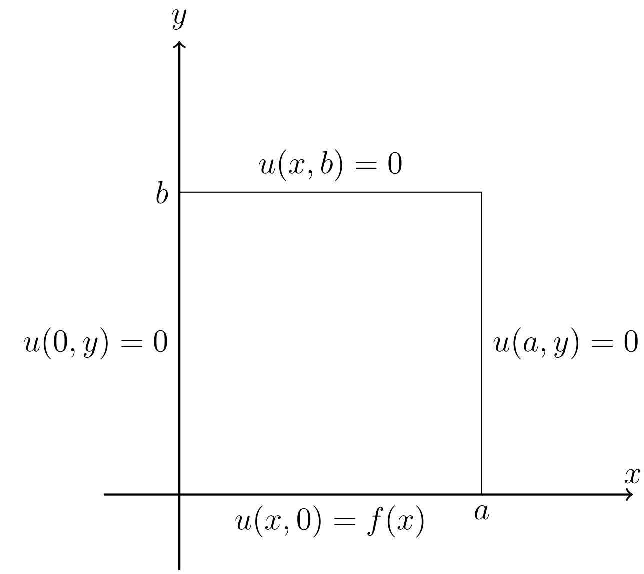
$$u_{xx} + u_{xx} = 0$$

This is an elliptic partial differential equation, which can be written with the shorthand $\triangle u = 0$. In polar coordinates, this can be written as:

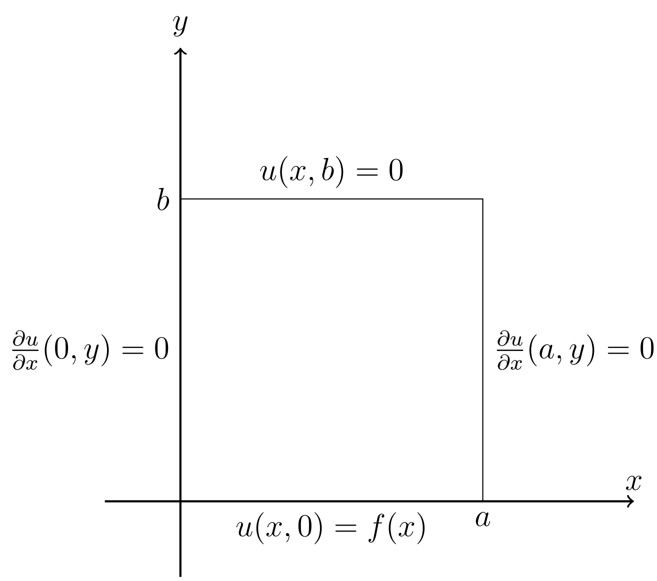
$$\triangle u = \nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$$

Boundary Conditions

Different boundary conditions are possible if $u(x, y)$ is a steady state temperature distribution on a rectangular plate. We can have homogeneous Dirichlet boundary conditions on the vertical and horizontal edges.



If we insulate the left and right edges of the plate, we can also have homogeneous Neumann boundary conditions.



Solution

The approach to solving this is still generally the same. Suppose we have the boundary value problem:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ u_x(0, y) &= u_x(a, y) = 0 \\ u(x, b) &= 0 \\ u(x, 0) &= f(x) \end{aligned}$$

We start with the separation of variables by substituting $u(x, y) = X(x)Y(y)$ to obtain two ordinary differential equations.

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \\ X''Y + XY'' &= 0 \\ \frac{X''}{X} &= \frac{-Y''}{Y} = -\lambda = -\alpha^2 \\ X'' + \lambda X &= 0 \\ Y'' - \lambda Y &= 0 \end{aligned}$$

We can then solve the eigenvalue problem by going through all three cases of λ. Since we have done this before, we know the following:

$$\begin{aligned} \lambda_n &= \left(\frac{n\pi}{a}\right)^2 \quad n = 0, 1, 2, \ldots \\ X_n(x) &= c_n \cos\left(\frac{n\pi x}{a}\right) \end{aligned}$$

With λ, we can solve the other ordinary differential equation.

$$\begin{aligned}
Y'' - \lambda Y &= 0 \\
\lambda_0 &= 0 \\
X_0 &= c_0 \\
Y_0(y) &= A_0 + B_0 y \\
\lambda_n &= \left(\frac{n\pi}{a}\right)^2 \\
Y_n(y) &= a_n \mathrm{e}^{\sqrt{\lambda}ny} + b_n \mathrm{e}^{-\sqrt{\lambda}ny} \\
&= a_n \cosh(\frac{n\pi y}{a}) + b_n \sinh(\frac{n\pi y}{a}) \\
u(x,b) &= X(x)Y(b) = 0 \\
Y(b) &= 0 \\
a_n &= -b_n \tanh(\frac{n\pi b}{a}) \\
Y_n(y) &= -b_n \tanh(\frac{n\pi b}{a}) \cosh(\frac{n\pi y}{a}) + b_n \sinh(\frac{n\pi y}{a}) \\
&= d_n \sinh(\frac{n\pi}{a}(y-b))
\end{aligned}$$

Now we can substitute our solutions for both ordinary differential equations back to get $u(x,y)$.

$$\begin{aligned}
u_n(x,y) &= e_n \cos(\frac{n\pi x}{a}) \sinh(\frac{n\pi}{a}(y-b)) \\
u(x,y) &= X_0 Y_0 + \sum X_n Y_n \\
&= e_0(y-b) + \sum_{n=1}^{\infty} e_n \cos(\frac{n\pi x}{a}) \sinh(\frac{n\pi}{a}(y-b)) \\
e_0 &= -\frac{1}{ab} \int_0^a f(x)\ \mathrm{d}x \\
e_n &= \frac{2}{a \sinh(\frac{-n\pi b}{a})} \int_0^a f(x) \cos(\frac{n\pi x}{a})\ \mathrm{d}x
\end{aligned}$$

Polar Coordinates

In polar coordinates, the Laplacian of $u(r,\theta)$ is

$$\triangle u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{2}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Suppose we have the following Dirichlet boundary value problem on a 2D disk shaped domain.

$$\begin{aligned}
u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} &= 0 \quad 0 \leq r < a \quad -\pi \leq \theta \leq \pi \\
u(r, -\pi) &= u(r, \pi) \quad 0 \leq r < a \\
u_\theta(r, -\pi) &= u_\theta(r, \pi) \quad 0 \leq r < a \\
u(a, \theta) &= f(\theta) \quad -\pi < \theta < \pi
\end{aligned}$$

We have to impose the additional constraint that $u(r, \theta)$ is bounded on the domain. Like before, we begin with separation of variables.

$$u(r, \theta) = R(r)T(\theta)$$

This yields the following ordinary differential equations and boundary conditions.

$$\begin{aligned}
T''(\theta) + \lambda T'(\theta) &= 0 \\
T(\pi) &= T(\pi) \\
T'(-\pi) &= T'(\pi) \\
r^2 R''(r) + rR'(r) - \lambda R(r) &= 0
\end{aligned}$$

The solution to the eigenvalue problem presented here is

$$\begin{aligned}
\lambda_n &= n^2 \quad n = 0, 1, 2, \ldots \\
T_0(\theta) &= B_0 \\
T_n(\theta) &= A_n \cos(n\theta) + B_n \sin(n\theta) \quad n = 1, 2, \ldots
\end{aligned}$$

The other ordinary differential equation ($r^2 R''(r) + rR'(r) - \lambda R(r) = 0$) is known as a Cauchy-Euler equation. We cannot use auxiliary equations to solve this because of the variable coefficients.

Case $n = 0$:

Suppose we take the case of λ_0:

$$\begin{aligned}
\lambda_0 &= 0 \\
r^2 R'' + rR' &= 0 \\
Let: y &= \frac{\mathrm{d}R}{\mathrm{d}r} = R' \\
r^2 y' + ry &= 0 \\
ry' + y &= 0 \\
\frac{\mathrm{d}y}{\mathrm{d}r} &= -\frac{y}{r} \\
\frac{\mathrm{d}y}{y} &= -\frac{\mathrm{d}r}{r} \\
y = R' &= \mathrm{e}^{-\ln(r)+c} \\
&= C\mathrm{e}^{-\ln(r)} = \frac{C}{r} \\
\frac{\mathrm{d}R}{\mathrm{d}r} &= \frac{C}{r} \\
\int \mathrm{d}R &= \int \frac{C}{r}\, \mathrm{d}r \\
R_0(r) &= C\ln(r) + d
\end{aligned}$$

Since we want a reasonable bounded solution, we can apply the constraint $|u(r, \theta)| < \infty$ for $0 \leq r \leq a$ and $-\pi \leq \theta < \pi$. Since $\ln(r)$ approaches $-\infty$ at 0, we set D_0 to 0 since it doesn't make sense for our circular disk to have infinite temperature at its center.

$$R_0(r) = C_0$$

Case $n > 0$:

Suppose we take $n = 1, 2, 3, \dots (n \in \mathbb{N})$. We can use a test solution $R(r) = r^m$ to try and obtain an m that satifies the differential equation.

$$\begin{aligned} r^2 R'' + rR' - n^2 R &= 0 \\ R(r) &= r^m \\ R'(r) &= mr^{m-1} \\ R''(R) &= m(m-1)r^{m-2} \\ r^2(m(m-1)r^{m-2}) + r(mr^{m-1}) - n^2 r^m &= 0 \\ (m(m-1) + m - n^2)r^m &= 0 \\ m^2 - m + m - n^2 &= 0 \\ m^2 &= n^2 \\ m &= \pm n \end{aligned}$$

The general solution can be put in the form:

$$R(r) = C_n r^n + D_n r^{-n}$$

If $R(r)$ is bounded, then we must set $D_n = 0$, otherwise r^{-n} goes to infinity as r goes to 0, which also does not make sense in the context of the problem.

$$R(r) = C_n r^n$$

With this, we can assemble the modes and construct our solution for $u_n(r, \theta)$. For $n = 0$:

$$\begin{aligned} u_0(r, \theta) &= R_0(r) T_0(\theta) \\ &= C_0 B_0 \\ &= \frac{a_0}{2} \quad \text{renamed for convenience} \end{aligned}$$

For all other $n \in \mathbb{N}$:

$$\begin{aligned} u_n(r, \theta) &= R_n T_n \\ &= C_n r^n (A_n \cos(n\theta) + B_n \sin(n\theta) \end{aligned}$$

We rename variables to put this in the form of a Fourier Series solution:

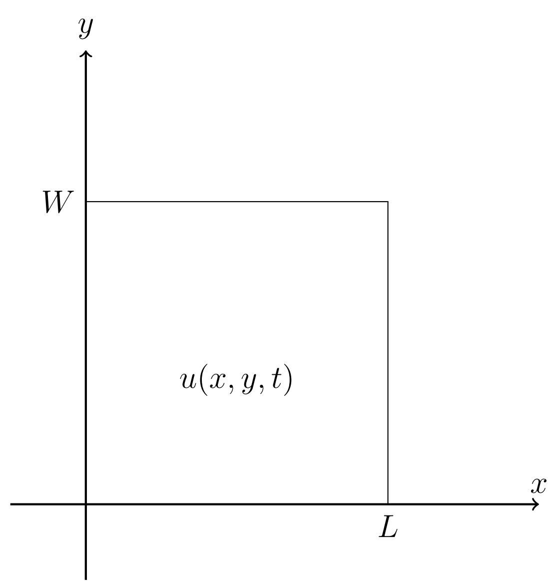
$$C_nA_n = \frac{a_n}{a^n}$$
$$C_nB_n = \frac{b_n}{a^n}$$
$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(\frac{r}{a})^n(a_n\cos(n\theta) + b_n\sin(n\theta))$$

If we apply the final boundary condition:

$$u(a,\theta) = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(\frac{a}{a})^n a_n\cos(n\theta) + b_n\sin(n\theta)$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\theta) + b_n\sin(n\theta)$$

This matches the form of a standard Fourier series solution.

Heat Equation in 2D



If $u(x, y, t)$ is 0 on all four boundaries and defined by the initial temperature $u(x, y, 0) = f(0)$, then we have the following boundary value problem.

$$\begin{aligned}
u_t &= \beta \triangle u = \beta(u_{xx} + u_{yy}) \\
u(0, y, t) &= 0 \\
u(L, y, t) &= 0 \\
u(x, 0, t) &= 0 \\
u(x, W, t) &= 0 \\
u(x, y, 0) &= f(x)
\end{aligned}$$

Again, we begin with separation of variables obtain ordinary differential equations. We will have to employ it twice here to obtain three ordinary differential equations.

$$\begin{aligned}
u(x, y, t) &= X(x)Y(y)T(t) \\
u_t &= \beta(u_{xx} + u_{yy}) \\
XYT' &- \beta(X''YT + XY''T) \\
\frac{T'}{\beta T} &= \frac{X''}{X} + \frac{Y''}{Y} = -\lambda \\
T' + \beta\lambda T &= 0 \\
\frac{X''}{X} + \frac{Y''}{Y} &= -\lambda \\
\frac{X''}{X} &= -\mu \quad X'' + \mu X = 0 \\
-\frac{Y''}{Y} - \lambda &= -\mu \quad Y'' + (\lambda - \mu)Y = 0
\end{aligned}$$

If we substitute and solve for the boundary conditions for our ordinary differential equations:

$$\begin{aligned}
X(0) &= X(L) = 0 \\
Y(0) &= Y(W) = 0
\end{aligned}$$

We've solved these eigenproblems before:

$$\begin{aligned}
\mu_m &= (\frac{m\pi}{L})^2 \quad m \in \mathbb{N} \\
X_m(x) &= c_m \sin(\frac{m\pi x}{L}) \\
Let: \epsilon &= \lambda - \mu \\
\epsilon_n &= (\frac{n\pi}{W})^2 \quad n \in \mathbb{N} \\
Y_n(y) &= a_n \sin(\frac{n\pi y}{W}) \\
\lambda_{mn} &= (\frac{n\pi}{W})^2 + (\frac{m\pi}{L})^2 \\
T' &= -\beta\lambda T \\
T(t) &= b\mathrm{e}^{-\beta\lambda t} \\
T_{mn}(t) &= b_{mn}\mathrm{e}^{-\beta(\frac{m^2}{L^2}+\frac{n^2}{W^2})\pi^2 t}
\end{aligned}$$

We can add the modes into a superposition as the solution and apply the initial condition.

$$\begin{aligned}
u_{mn}(x,y,t) &= X_m(x)Y_n(y)T_{mn}(t) \\
u(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} u_{mn}(x,y,t) \\
u(x,y,0) &= f(x,y) \\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin(\frac{m\pi x}{L})\sin(\frac{n\pi y}{W})
\end{aligned}$$

This takes the form of a Double Fourier Series solution. If we solve for a_{mn} using the Double Fourier Series formulas:

$$a_{mn} = \frac{4}{LW}\int_0^L\int_0^W f(x,y)\sin(\frac{m\pi x}{L})\sin(\frac{n\pi y}{W})\ \mathrm{d}x\ \mathrm{d}y$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech