Boundary Value Problems

Alvin Lin

August 2018 - December 2018

Wave Equation in 1D

Suppose have a string stretched between two endpoints $x = 0$ and $x = L$. $y(x, t)$ is the transverse displacement from $y = 0$, the resting or equilibrium position. At rest, $y = 0$ and y_t, the velocity of the string, is also 0. If we take the following assumptions:

- no gravity
- string is perfectly flexible
- string has uniform properties (e.g. density)
- $|y| << L$ and derivatives of y remain relatively small
- a uniform tension T is exerted on the string
- no other external forces

We can develop a partial differential equation using force balance for a segment of the string. If we apply Newton's second law, letting $\delta x \to 0$, we obtain the following equation.

$$y_{tt} = a^2 y_{xx}$$

where $a^2 = \frac{|T|}{\rho} > 0$ and ρ is the string density. a is analogous to acceleration in the formula $a = \frac{F}{m}$. This is the wave equation in 1 dimension, but we will need two boundary conditions and two initial conditions (unlike the heat equation which needed one). Once we have those, it can be used to model longitudinal waves and water waves in two dimensions or higher.

Boundary Conditions

Suppose at one end of the string $x = 0$, the string is fixed to a pole. This gives us the boundary condition $y(0,t) = 0$, a homogeneous Dirichlet boundary condition.

Suppose instead that the string is attached to a ring that freely slides up and down the pole, this would instead give us the boundary condition $\frac{\partial y}{\partial x}(0,t) = 0$ or $y_x(0,t) = 0$, which is a homogeneous Neumann boundary condition. Thus we have the partial differential equation:

$$\begin{aligned} y_{tt} &= a^2 y_{xx} \\ y(0,t) &= y(L,t) = 0 \\ y(x,0) &= f(x) \quad \text{(some initial displacement)} \\ y_t(x,0) &= g(x) \quad \text{(some initial velocity)} \end{aligned}$$

Process

Solving this follows the same process:

1. Use separation of variables by substituting $y = X(x)T(t)$.
2. Solve the X based eigenvalue/eigenfunction problem to get some eigenvalue λ_n and its corresponding eigenfunction $X_n(x)$.
3. Solve the T based ordinary differential equation using λ_n.
4. Substitute back to get an equation for the modes.

   $$y_n(x,t) = X_n(x)T_n(t)$$

   Use superposition to produce a formal solution.

   $$y(x,t) = \sum_n y_n(x,t)$$

   where y_n contains some arbitrary constants dependent on n.
5. Apply the initial conditions. Use Fourier series methods to solve for the arbitrary constants and plug it back into $y(x,t)$.

Example

$$
\begin{aligned}
y &= X(x)T(t) \\
XT_{tt} &= a^2 X_{xx} T \\
\frac{T_{tt}}{a^2 T} &= \frac{X_{xx}}{X} = \lambda \\
X_n(x) &= c_n \sin(\frac{n\pi x}{L}) \\
\lambda_n &= \left(\frac{n\pi}{L}\right)^2 \\
T'' + \lambda a^2 T &= 0 \\
r^2 + \lambda a^2 &= 0 \\
r &= -\sqrt{\lambda} a \\
T(t) &= c_1 \cos(a\sqrt{\lambda} t) + c_2 \sin(a\sqrt{\lambda} t) \\
&= A_n \cos(a\frac{n\pi}{L} t) + B_n \sin(a\frac{n\pi}{L} t) \\
y_n &= X_n(x) T_n(t) \\
y(x,t) &= \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L})(A_n \cos(a\frac{n\pi}{L} t) + B_n \sin(a\frac{n\pi}{L} t)) \\
a_n &:= c_n A_n \quad b_n := c_n B_n \\
y(x,t) &= \sum_{n=1}^{\infty} sin(\frac{n\pi x}{L})(a_n \cos(\frac{an\pi}{L} t) + b_n \sin(\frac{an\pi}{L} t))
\end{aligned}
$$

At this point, we can apply the initial conditions:

$$
\begin{aligned}
y(x,0) &= f(x) \\
&= \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L})(a_n \cos(0) + b_n \sin(0)) \\
&= \sum_{n=1}^{\infty} a_n \sin(\frac{n\pi x}{L}) \\
a_n &= \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\ \mathrm{d}x \\
y_t(x,0) &= g(x) \\
\frac{\partial y}{\partial t} &= \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L})\frac{an\pi}{L}\left(-a_n \sin(\frac{an\pi}{L}t) + b_n \cos(\frac{an\pi}{L}t)\right) \\
\frac{\partial y}{\partial t}(0) &= \sum_{n=1}^{\infty} \frac{an\pi}{L}\sin(\frac{n\pi x}{L})(-a_n \sin(0) + b_n \cos(0)) \\
&= \sum_{n=1}^{\infty} b_n(\frac{an\pi}{L})\sin(\frac{n\pi x}{L}) = g(x) \\
b_n \frac{an\pi}{L} &= \frac{2}{L}\int_0^L g(x)\sin(\frac{n\pi x}{L})\ \mathrm{d}x \\
b_n &= \frac{2}{an\pi}\int_0^L g(x)\sin(\frac{n\pi x}{L})\ \mathrm{d}x
\end{aligned}
$$

D'Alembert's Method

Assume an infinitely long string:

$$
\begin{aligned}
y_{tt} &= a^2 y_{xx} \quad -\infty < x < \infty \quad t > 0 \\
y(x,0) &= f(x) \quad -\infty < x < \infty \\
y_t(x,0) &= g(x) \quad -\infty < x < \infty
\end{aligned}
$$

D'Alembert's solution gives an equation of the form:

$$
y(x,t) = f_1(x+at) + f_2(x-at)
$$

We can verify that it's a solution by substituting it into the partial differential equation. First find y_{tt} in terms of f_1 and f_2.

$$\begin{aligned}
y_t &= \frac{\partial}{\partial t}(f_1(x+at) + f_2(x-at)) \\
&= f_1'(x+at)\frac{\partial}{\partial t}(x+at) + f_2'(x-at)\frac{\partial}{\partial t}(x-at) \\
&= af_1'(x+at) - af_2'(x-at) \\
y_{tt} &= a^2 f_1''(x+at) + a^2 f_2''(x-at)
\end{aligned}$$

Next, find y_{xx} in terms of f_1 and f_2 and substitute it into the equation above to obtain the original partial differential equation.

$$\begin{aligned}
y_{xx} &= \frac{\partial^2}{\partial x^2}(f_1(x+at) + f_2(x-at)) \\
&= f_1''(x+at) + f_2''(x-at) \\
y_{tt} &= a^2 y_{xx}
\end{aligned}$$

Example

Find a solution to

$$\begin{aligned}
y_{tt} &= 100 y_{xx} \quad -\infty < x < \infty \\
y(x,0) &= \cos^2(x) \\
y_t(x,0) &= 2x \\
a &= \sqrt{100} = 10 \quad f(x) = \cos^2(x) \quad g(x) = 2x \\
y(x,t) &= f_1(x+10t) + f_2(x-10t) \\
y(x,0) &= f_1(x) + f_2(x) = \cos^2(x) \\
y_t(x,0) &= g(x) = 2x \\
&= 10f_1'(x+10t) - 10f_2'(x-10t) \\
y_t(x,0) &= 10f_1'(x) - 10f_2'(x) = 2x \\
f_1'(x) - f_2'(x) &= \frac{x}{5} \\
\int (f_1'(x) - f_2'(x)) \, \mathrm{d}x &= \int \frac{x}{5} \, \mathrm{d}x + c \\
f_1(x) + f_2(x) &= \frac{x^2}{10} + c
\end{aligned}$$

We can solve the system of equations for f_1 and f_2:

$$\begin{aligned}
f_1(x) &= \frac{\cos^2(x)}{2} + \frac{x^2}{20} + \frac{c}{2} \\
f_2(x) &= \frac{\cos^2(x)}{2} - \frac{x^2}{20} - \frac{c}{2} \\
y(x,t) &= \frac{\cos^2(x+10t)}{2} + \frac{(x+10t)^2}{20} + \frac{\cos^2(x-10t)}{2} + \frac{(x-10t)^2}{20}
\end{aligned}$$

Standing vs Traveling Waves

The solution of a Dirichlet 1D homogeneous boundary value problem such as

$$y_{tt} = a^2 y_{xx}$$

is of the form:

$$y(x,t) = \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi at}{L}) + b_n \sin(\frac{n\pi at}{L})) \sin(\frac{n\pi x}{L})$$

This solution represents a superposition of standing waves with the following properties:

- Mode amplitude
$$a_n \cos(\frac{n\pi at}{L}) + b_n \sin(\frac{n\pi at}{L})$$
- Mode shape
$$\sin(\frac{n\pi x}{L})$$

Traveling waves showed up in D'Alembert's solution in where the solution

$$y = f_1(x+at) + f_2(x-at)$$

represents translating f_1 and f_2 horizontally. Now consider a mode of a Dirichlet solution:

$$y_n(x,t) = \sin(\frac{n\pi at}{L}) \sin(\frac{n\pi x}{L})$$

Can we rewrite this in terms of traveling waves? By using a trigonometric identity:

$$\begin{aligned}
\sin(\frac{n\pi at}{L}) \sin(\frac{n\pi x}{L}) &= \sin(A)\sin(B) \\
&= \frac{1}{2}(\cos(A-B) - \cos(A+B)) \\
&= \frac{1}{2}(\cos(\frac{n\pi}{L}(at-x)) - \cos(\frac{n\pi}{L}(at+x)))
\end{aligned}$$

Thus, a standing wave is a superposition of traveling waves.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech