Boundary Value Problems

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Nonhomogeneous Heat Equation Solutions

An example of a homogeneous heat equation is

$$\begin{aligned} u_t &= a^2 u_{xx} \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= f(x) \end{aligned}$$

To make the equation nonhomogeneous:

$$\begin{aligned} u_t &= a^2 u_{xx} + P(x) \\ u(0,t) &= u_1 \\ u(L,t) &= u_2 \\ u(x,0) &= f(x) \end{aligned}$$

where $P(x)$ is a heat source or sink. If $u_1 \neq 0$ or $u_2 \neq 0$ or $P(x) \neq 0$ then the boundary value problem is nonhomogeneous. The simpler case of this involves $u_1 = u_2 \neq 0$, in which case we will define $v(x,t) = u(x,t) - u_1$, which can substitute back into the boundary value problem. The harder case involves $u_1 \neq u_2$ or $P(x) \neq 0$.

Methods for Solving

- Substitute $u(x,t) = \psi(x) + v(x,t)$ into the partial differential equation, boundary conditions, and initial condition.
- Solve the ordinary differential equation for $\psi(x)$.
- Solve the homogeneous partial differential equation for $v(x,t)$.
- Substitute back into $u(x,t) = \psi(x) + v(x,t)$.

Interpretation

$$u(x,t) = \psi(x) + v(x,t)$$

$\psi(x)$ represents a steady-state solution that satisfies nonhomogeneous terms while $v(x,t)$ is a transient solution that dies off as t approaches infinity.

Example

$$\begin{aligned}
u_t &= u_{xx} \quad 0 < x < \pi \\
u(0,t) &= 0 \\
u(\pi,t) &= 3\pi \\
u(x,0) &= f(x) = 0
\end{aligned}$$

We want to find a solution of the form $u(x,t) = \psi(x) + v(x,t)$ so we will substitute this into the partial differential equation.

$$\begin{aligned}
\frac{\partial}{\partial t}(u) = \frac{\partial}{\partial t}(\psi(x) + v(x,t)) &= a^2 \frac{\partial^2}{\partial x^2}(\psi(x) + v(x,t)) \\
\frac{\partial \psi(x)}{\partial t} + v_t(x,t) &= a^2(\psi''(x) + v_{xx}(x,t)) \\
v_t(x,t) &= a^2(\psi''(x) + v_{xx}(x,t)) \\
u(0,t) &= \psi(0) + v(0,t) = u_1 \\
u(L,t) &= \psi(L) + v(L,t) = u_2 \\
u(x,0) &= \psi(x) + v(x,0) = f(x)
\end{aligned}$$

Let $t \to \infty$ to obtain the steady state boundary value problem:

$$\begin{aligned}
v_t &= a^2(\psi'' + v_x x) \\
0 &= a^2 \psi'' \\
\psi(0) + v(0,t) &= u_1 \\
\psi(0) &= u_1 \\
\psi(L) + v(L,t) &= u_2 \\
\psi(L) &= u_2
\end{aligned}$$

We now have a steady state boundary value problem which we can solve:

$$\begin{aligned}
a^2\psi''(x) &= 0 \quad (a^2 > 0)\\
\psi(0) &= u_1\\
\psi(L) &= u_2\\
\psi(x) &= Ax + B\\
\psi(0) &= u_1 = A(0) + B = B\\
u_1 &= B\\
\psi(L) &= u_2 = AL + B = AL + u_1\\
AL &= u_2 - u_1\\
A &= \frac{u_2 - u_1}{L}\\
\psi(x) &= (\frac{u_2 - u_1}{L})x + u_1
\end{aligned}$$

We can plug $\psi(x)$ back into the boundary value problem to solve for $v(x,t)$ for arbitrary $t > 0$.

$$\begin{aligned}
v_t &= a^2(\psi'' + v_{xx})\\
\psi(0) + v(0,t) &= u_1\\
u_1 + v(0,t) &= u_1\\
v(0,t) &= 0\\
\psi(L) + v(L,t) &= u_2\\
u_2 + v(L,t) &= u_2\\
v(L,t) &= 0\\
u(x,0) &= v(x,0) + \psi(x) = f(x)\\
v(x,0) &= f(x) - \psi(x)
\end{aligned}$$

We can now solve the homogeneous boundary value problem for $v(x,t)$.

$$\begin{aligned}
v_t &= a^2 v_{xx}\\
v(0,t) &= v(L,t) = 0\\
v(x,0) &= f(x) - \psi(x)\\
&= f(x) - (\frac{u_2 - u_1}{L})x + u_1
\end{aligned}$$

Note that this homogeneous boundary value problem is in the same form as the Dirichlet boundary value problem we have already solved.

$$
\begin{aligned}
v(x,t) &= \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) \mathrm{e}^{-a^2(\frac{n\pi}{L})^2 t} \\
b_n &= \frac{2}{L} \int_0^L (f(x) - (\frac{u_2 - u_1}{L}) - u_1) \sin(\frac{n\pi x}{L}) \, \mathrm{d}x \\
u(x,t) &= \psi(x) + v(x,t) \\
&= (\frac{u_2 - u_1}{L})x + u_1 + v(x,t)
\end{aligned}
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