

Boundary Value Problems

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Nonhomogeneous Heat Equation Solutions

An example of a homogeneous heat equation is

$$\begin{aligned}u_t &= a^2 u_{xx} \\ u(0, t) &= u(L, t) = 0 \\ u(x, 0) &= f(x)\end{aligned}$$

To make the equation nonhomogeneous:

$$\begin{aligned}u_t &= a^2 u_{xx} + P(x) \\ u(0, t) &= u_1 \\ u(L, t) &= u_2 \\ u(x, 0) &= f(x)\end{aligned}$$

where $P(x)$ is a heat source or sink. If $u_1 \neq 0$ or $u_2 \neq 0$ or $P(x) \neq 0$ then the boundary value problem is nonhomogeneous. The simpler case of this involves $u_1 = u_2 \neq 0$, in which case we will define $v(x, t) = u(x, t) - u_1$, which can substitute back into the boundary value problem. The harder case involves $u_1 \neq u_2$ or $P(x) \neq 0$.

Methods for Solving

- Substitute $u(x, t) = \psi(x) + v(x, t)$ into the partial differential equation, boundary conditions, and initial condition.
- Solve the ordinary differential equation for $\psi(x)$.
- Solve the homogeneous partial differential equation for $v(x, t)$.
- Substitute back into $u(x, t) = \psi(x) + v(x, t)$.

Interpretation

$$u(x, t) = \psi(x) + v(x, t)$$

$\psi(x)$ represents a steady-state solution that satisfies nonhomogeneous terms while $v(x, t)$ is a transient solution that dies off as t approaches infinity.

Example

$$\begin{aligned}u_t &= u_{xx} & 0 < x < \pi \\u(0, t) &= 0 \\u(\pi, t) &= 3\pi \\u(x, 0) &= f(x) = 0\end{aligned}$$

We want to find a solution of the form $u(x, t) = \psi(x) + v(x, t)$ so we will substitute this into the partial differential equation.

$$\begin{aligned}\frac{\partial}{\partial t}(u) &= \frac{\partial}{\partial t}(\psi(x) + v(x, t)) = a^2 \frac{\partial^2}{\partial x^2}(\psi(x) + v(x, t)) \\ \frac{\partial \psi(x)}{\partial t} + v_t(x, t) &= a^2(\psi''(x) + v_{xx}(x, t)) \\ v_t(x, t) &= a^2(\psi''(x) + v_{xx}(x, t)) \\ u(0, t) &= \psi(0) + v(0, t) = u_1 \\ u(L, t) &= \psi(L) + v(L, t) = u_2 \\ u(x, 0) &= \psi(x) + v(x, 0) = f(x)\end{aligned}$$

Let $t \rightarrow \infty$ to obtain the steady state boundary value problem:

$$\begin{aligned}v_t &= a^2(\psi'' + v_{xx}) \\ 0 &= a^2\psi'' \\ \psi(0) + v(0, t) &= u_1 \\ \psi(0) &= u_1 \\ \psi(L) + v(L, t) &= u_2 \\ \psi(L) &= u_2\end{aligned}$$

We now have a steady state boundary value problem which we can solve:

$$\begin{aligned}
 a^2\psi''(x) &= 0 \quad (a^2 > 0) \\
 \psi(0) &= u_1 \\
 \psi(L) &= u_2 \\
 \psi(x) &= Ax + B \\
 \psi(0) = u_1 &= A(0) + B = B \\
 u_1 &= B \\
 \psi(L) = u_2 &= AL + B = AL + u_1 \\
 AL &= u_2 - u_1 \\
 A &= \frac{u_2 - u_1}{L} \\
 \psi(x) &= \left(\frac{u_2 - u_1}{L}\right)x + u_1
 \end{aligned}$$

We can plug $\psi(x)$ back into the boundary value problem to solve for $v(x, t)$ for arbitrary $t > 0$.

$$\begin{aligned}
 v_t &= a^2(\psi'' + v_{xx}) \\
 \psi(0) + v(0, t) &= u_1 \\
 u_1 + v(0, t) &= u_1 \\
 v(0, t) &= 0 \\
 \psi(L) + v(L, t) &= u_2 \\
 u_2 + v(L, t) &= u_2 \\
 v(L, t) &= 0 \\
 u(x, 0) = v(x, 0) + \psi(x) &= f(x) \\
 v(x, 0) &= f(x) - \psi(x)
 \end{aligned}$$

We can now solve the homogeneous boundary value problem for $v(x, t)$.

$$\begin{aligned}
 v_t &= a^2v_{xx} \\
 v(0, t) = v(L, t) &= 0 \\
 v(x, 0) &= f(x) - \psi(x) \\
 &= f(x) - \left(\frac{u_2 - u_1}{L}\right)x + u_1
 \end{aligned}$$

Note that this homogeneous boundary value problem is in the same form as the Dirichlet boundary value problem we have already solved.

$$\begin{aligned}v(x, t) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-a^2\left(\frac{n\pi}{L}\right)^2 t} \\b_n &= \frac{2}{L} \int_0^L \left(f(x) - \left(\frac{u_2 - u_1}{L}\right)x - u_1\right) \sin\left(\frac{n\pi x}{L}\right) dx \\u(x, t) &= \psi(x) + v(x, t) \\&= \left(\frac{u_2 - u_1}{L}\right)x + u_1 + v(x, t)\end{aligned}$$

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