

Boundary Value Problems

Alvin Lin

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Boundary Condition Types

	In terms of $X(x)$	In terms of $u(x, t)$
Dirichlet	$\begin{aligned} X(a) &= c_1 \\ X(b) &= c_2 \end{aligned}$	$\begin{aligned} u(a, t) &= c_1 \\ u(b, t) &= c_2 \end{aligned}$
Neumann	$\begin{aligned} X'(a) &= c_1 \\ X'(b) &= c_2 \end{aligned}$	$\begin{aligned} u_x(a, t) &= c_1 \\ u_x(b, t) &= c_2 \end{aligned}$
Separated	$\begin{aligned} a_1 X(a) + a_2 X'(a) &= c_1 \\ b_1 X(b) + b_2 X'(b) &= c_2 \end{aligned}$	$\begin{aligned} a_1 u(a, t) + a_2 u_x(a, t) &= c_1 \\ b_1 u(b, t) + b_2 u_x(b, t) &= c_2 \end{aligned}$
Periodic	$\begin{aligned} X(a) &= X(b) \\ X'(a) &= X'(b) \end{aligned}$	$\begin{aligned} u(a, t) &= u(b, t) \\ u_x(a, t) &= u_x(b, t) \end{aligned}$

Dirichlet, Neumann, and Robin boundary conditions are homogeneous if $c_1 = c_2 = 0$ and nonhomogeneous if $c_1 \neq 0$ or $c_2 \neq 0$. A combination of these can result however, for example:

$$\begin{aligned}u(0, t) &= 0 && \text{homogeneous Dirichlet} \\u_x(L, t) &= 1 && \text{nonhomogeneous Neumann}\end{aligned}$$

Boundary conditions are usually considered separately for homogeneity.

Example

In a previous section, we solved

$$\begin{aligned}u_t &= \beta u_x \\u(0, t) &= u(L, t) = 0 \\u(x, 0) &= f(x)\end{aligned}$$

We arrived at the solution:

$$\begin{aligned}u(x, t) &= \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \\b_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx\end{aligned}$$

The individual terms when n is chosen are referred to as modes:

$$\begin{aligned}u_n(x, t) &= \sin\left(\frac{n\pi x}{L}\right) e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \\u_1(x, t) &= \sin\left(\frac{\pi x}{L}\right) e^{-\beta\frac{\pi^2}{L^2} t} \\u_2(x, t) &= \sin\left(\frac{2\pi x}{L}\right) e^{-\beta\frac{2\pi^2}{L^2} t}\end{aligned}$$

Due to the exponential term, the modes u_n approach 0 as t approaches infinity.

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech