Boundary Value Problems

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Boundary Condition Types

	In terms of $X(x)$	In terms of $u(x,t)$
Dirichlet	$X(a) = c_1$ $X(b) = c_2$	$u(a,t) = c_1$ $u(b,t) = c_2$
Neumann	$X'(a) = c_1$ $X'(b) = c_2$	$u_x(a,t) = c_1$ $u_x(b,t) = c_2$
Separated	$a_1X(a) + a_2X'(a) = c_1$ $b_1X(b) + b_2X'(b) = c_2$	$a_1u(a,t) + a_2u_x(a,t) = c_1$ $b_1u(b,t) + b_2u_x(b,t) = c_2$
Periodic	$X(a) = X(b)$ $X'(a) = X'(b)$	$u(a,t) = u(b,t)$ $u_x(a,t) = u_x(b,t)$

Dirichlet, Neumann, and Robin boundary conditions are homogeneous if $c_1 = c_2 = 0$ and nonhomogeneous if $c_1 \neq 0$ or $c_2 \neq 0$. A combination of these can result however, for example:

$$\begin{aligned} u(0,t) &= 0 \quad \text{homogeneous Dirichlet} \\ u_x(L,t) &= 1 \quad \text{nonhomogeneous Neumann} \end{aligned}$$

Boundary conditions are usually considered separately for homogeneity.

Example

In a previous section, we solved

$$\begin{aligned} u_t &= \beta u_x x \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= f(x) \end{aligned}$$

We arrived at the solution:

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L}) \mathrm{e}^{-\beta(\frac{n\pi}{L})^2 t} \\ b_n &= \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\ \mathrm{d}x \end{aligned}$$

The individual terms when n is chosen are referred to as modes:

$$\begin{aligned} u_n(x,t) &= \sin(\frac{n\pi x}{L})\mathrm{e}^{-\beta(\frac{n\pi}{L}^2 t)} \\ u_1(x,t) &= \sin(\frac{\pi x}{L})\mathrm{e}^{-\beta\frac{\pi}{L}t} \\ u_2(x,t) &= \sin(\frac{2\pi x}{L})\mathrm{e}^{-\beta\frac{2\pi}{L}t} \end{aligned}$$

Due to the exponential term, the modes u_n approach 0 as t approaches infinity.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech