

# Boundary Value Problems

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August 2018 - December 2018

## Separation of Variables

Knowing if a partial differential equation is separable can help us solve it. Suppose we have the function  $u_t = 8u_{xx}$ . To test for separability:

1. Suppose  $u(x, t) = X(x)T(t)$ .
2. We need to substitute this into the partial differential equation.
3. If you can isolate all instances of the  $x$  independent variable on one side and the  $t$  independent variable on the other side of the equation, then the partial differential equation is separable.

$$\begin{aligned}u(x, t) &= X(x)T(t) \\ \frac{\partial}{\partial t}(X(x)T(t)) &= 8 \frac{\partial^2}{\partial x^2}(X(x)T(t)) \\ X(x) \frac{\partial}{\partial t}T(t) &= 8 \frac{\partial^2}{\partial x^2}(X(x))T(t) \\ X(x)T'(t) &= 8X''(x)T(t) \\ \frac{X(x)}{X''(x)} &= \frac{8T'(t)}{T(t)}\end{aligned}$$

In their separated form, we can set both sides equal to some separation constant to solve two ordinary differential equations. This separation constant is usually denoted  $\lambda$ ,  $-\lambda$ ,  $\alpha^2$ , or  $-\alpha^2$ .

$$\frac{X(x)}{X''(x)} = \frac{8T'(t)}{T(t)} = -\lambda$$

Separation is useful because we can break a partial differential equation into multiple ordinary differential equations.

### Example

Generic form of the heat equation:

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

We want to find a solution that is nontrivial and bounded. It helps to narrow things down if we have boundary conditions and/or initial conditions.

$$u(0, t) = 0 \quad u(L, t) = 0 \quad u(x, 0) = f(x)$$

We want to find  $u(x, t)$  that satisfies the partial differential equation, its boundary conditions, and its initial condition. Since the equation is separable, we can substitute  $u(x, t) = X(x)T(t)$  into the equation and show that

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{\beta T(t)}$$

If we choose the separation constant  $-\lambda$ :

$$\begin{aligned} \frac{X''(x)}{X(x)} &= -\lambda = \frac{T'(t)}{\beta T(t)} \\ X''(x) &= -\lambda X(x) \\ T'(t) &= -\lambda \beta T(t) \end{aligned}$$

With the boundary condition  $u(0, t) = 0$ , we know that  $X(0)T(t) = 0$ . With the boundary condition  $u(L, t) = 0$ , we know that  $X(L)T(t) = 0$ .

$$\begin{aligned} u(x, 0) &= X(x)T(0) = f(x) \\ X(x) &= \frac{f(x)}{T(0)} \end{aligned}$$

We can start with either ordinary differential equation:

$$\begin{aligned} X''(x) &= -\lambda X(x) \quad X(0) = 0 \quad X(L) = 0 \\ X''(x) + \lambda X(x) &= 0 \\ r^2 + \lambda &= 0 \end{aligned}$$

Using this characteristic equation, we have three cases:

1.  $\lambda < 0$

$$r^2 = -\lambda$$

$$r_{1,2} = \pm\sqrt{-\lambda}$$

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

$$X(0) = 0 = c_1 e^0 + c_2 e^0$$

$$c_1 = -c_2$$

$$X(L) = 0 = c_1 (e^{\sqrt{-\lambda}L} - e^{-\sqrt{-\lambda}L})$$

$$0 = c_1 (e^{2\sqrt{-\lambda}L} - 1)$$

$$c_1 = 0 = c_2$$

$$X(x) = 0 \quad (\text{trivial solution})$$

2.  $\lambda = 0$

$$X''(x) = 0$$

$$r^2 = 0$$

$$r = 0$$

$$X(x) = c_1 e^{0x} + c_2 x e^{0x}$$

$$= c_1 + x c_2$$

$$X(0) = 0 = c_1$$

$$c_1 = 0$$

$$X(L) = x c_2$$

$$c_2 = 0$$

$$X(x) = 0 \quad (\text{trivial solution})$$

3.  $\lambda > 0$

$$\begin{aligned}r^2 + \lambda &= 0 \\r_{1,2} &= \pm\sqrt{-\lambda} \\ \alpha^2 &= -\lambda \\ r_{1,2} &= \pm i\alpha = \pm i\sqrt{\lambda} \\ X(x) &= c_1 \cos(\alpha x) + c_2 \sin(\alpha x) \\ X(0) = 0 &= c_1 \cos(0) + c_2 \sin(0) \\ c_1 &= 0 \\ X(L) = 0 &= c_2 \sin(\alpha L)\end{aligned}$$

If  $c_2$  is 0, then the solution is again trivial. If it is not zero, then the solution depends on the separation constant.

$$\begin{aligned}\sin(\alpha L) &= 0 \\ \alpha L &= n\pi, \quad n \in \mathbb{Z} \\ \alpha &= \frac{n\pi}{L} \\ X(x) &= c_2 \sin\left(\frac{n\pi}{L}x\right)\end{aligned}$$

To produce a non-trivial solution for  $X(x)$ , we needed to use  $\sqrt{\lambda} = \frac{n\pi}{L}$ . We can use this to solve the second ordinary differential equation.

$$\begin{aligned}T'(t) &= -\lambda\beta T(t) \\ T'(t) &= -\frac{n^2\pi^2}{L^2}\beta T(t) \\ T(t) &= c_2 e^{ct} \\ T'(t) &= \frac{d}{dt}(T(t)) \\ &= c_2 c e^{ct} \\ &= cT(t) \\ T(t) &= c_3 e^{-\frac{n^2\pi^2}{L^2}\beta t}\end{aligned}$$

We can substitute this back into  $u(x, t) = X(x)T(t)$  to find our solution.

$$\begin{aligned}u(x, t) &= X(x)T(t) \\ &= c_2 \sin\left(\frac{n\pi x}{L}\right) c_3 e^{-\frac{n^2\pi^2}{L^2}\beta t} \\ u_n(x, t) &= c_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2\beta t}\end{aligned}$$

This has infinitely many solutions because haven't applied our initial condition yet.

## Eigenvalues and Eigenfunctions

Thus far, we have found special  $X(x)$  and  $\lambda$  values that made the boundary value problem solutions non-trivial. These are known as eigenfunctions:

$$\begin{aligned}X_n(x) &= c_n \sin\left(\frac{n\pi x}{L}\right) = \text{eigenfunctions of the BVP} \\ \lambda_n &= \left(\frac{n\pi}{L}\right)^2 = \text{eigenvalues of the BVP}\end{aligned}$$

We can associate eigenvalues and eigenfunctions with operators.

$$\begin{aligned}X'' + \lambda X &= 0 \\ X'' &= -\lambda X \\ LX &= -\lambda X \\ Ly &= D^2y = y'' \\ LX &= D^2X = X''\end{aligned}$$

Eigenvalues of a linear operator  $L$  are values for which  $Ly = \lambda y$  has non-trivial solutions for the given boundary conditions. Eigenfunctions are functions  $y_n(x)$  corresponding to eigenvalues  $\lambda_n$ . Problems that require you to find the eigenvalues and eigenfunctions of boundary value problems require you to find all non-trivial  $y_n$  solutions.

## Superposition Principle

If  $u_1, u_2, \dots, u_n$  are solutions to a linear homogeneous differential equation, then  $u(x, t) = \sum_{i=1}^{\infty} c_i u_i(x, t)$  is also a solution where  $c_i$  are arbitrary constants. If we look at the solution we acquired before:

$$u_n(x, t) = c_n \sin\left(\frac{n\pi x}{L}\right) e^{-(\frac{n\pi}{L})^2\beta t}$$

We can rewrite this as:

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 \beta t}$$

## Summary

To recap, these are the steps to solve this type of boundary value problem:

1. Separate the variables.

$$u(x, t) = X(x)T(t)$$

Plug these back into the partial differential equation to get two ordinary differential equations related by a separation constant  $\lambda$  or  $\alpha^2$ .

2. Use the boundary conditions to solve for  $X(x)$ . After separation, we have a second order differential equation  $X$ , which yields an auxiliary equation whose roots depend on the sign of  $\lambda$ . This yields a family of solutions  $X_1(x), X_2(x), \dots$  (eigenfunctions) corresponding to the  $\lambda$  values  $\lambda_1, \lambda_2, \dots$  (eigenvalues).
3. Find  $T_n(t)$  for each  $\lambda_n$ .
4. Solutions should look like  $u_n(x, t) = X_n(x)T_n(t)$ .
5. Use superposition and Fourier series (if needed) to find the rest of the arbitrary constants  $c_n$  that fit the initial conditions.

## Continuing the Previous Example

$$\begin{aligned}u_t &= \beta u_{xx} \\u(0, t) &= u(L, t) = 0 \\u(x, 0) &= f(x) = 1\end{aligned}$$

We can apply the initial condition starting from the solution we acquired earlier.

$$\begin{aligned}u(x, t) &= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 \beta t} \\u(x, 0) &= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) = 1 = f(x)\end{aligned}$$

This becomes a Fourier Sine Series problem with  $L = 1$  and  $b_n = c_n$ .

$$\begin{aligned}c_n &= \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \\&= 2 \int_0^1 1 \sin(n\pi x) dx \\&= \frac{2}{n\pi} (1 - \cos(n\pi))\end{aligned}$$

We can substitute this back into the formal solution.

$$\begin{aligned}u(x, t) &= \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{n\pi}{L}\right)^2 \beta t} \\&= \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (1 - \cos(n\pi)) \sin(n\pi x) e^{-\beta(n\pi)^2 t}\end{aligned}$$

This is a solution to the boundary value problem when  $f(x) = 1$  and  $L = 1$ .

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)