Boundary Value Problems

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August 2018 - December 2018

Separation of Variables

Knowing if a partial differential equation is separable can help us solve it. Suppose we have the function $u_t = 8u_{xx}$. To test for separability:

1. Suppose $u(x,t) = X(x)T(t)$.
2. We need to substitute this into the partial differential equation.
3. If you can isolate all instances of the x independent variable on one side and the t independent variable on the other side of the equation, then the partial differential equation is separable.

$$\begin{aligned}
u(x,t) &= X(x)T(t) \\
\frac{\partial}{\partial t}(X(x)T(t)) &= 8\frac{\partial^2}{\partial x^2}(X(x)T(t)) \\
X(x)\frac{\partial}{\partial t}T(t) &= 8\frac{\partial^2}{\partial x^2}(X(x))T(t) \\
X(x)T'(t) &= 8X''(x)T(t) \\
\frac{X(x)}{X''(x)} &= \frac{8T'(t)}{T(t)}
\end{aligned}$$

In their separated form, we can set both sides equal to some separation constant to solve two ordinary differential equations. This separation constant is usually denoted $\lambda, -\lambda, \alpha^2, \text{ or } -\alpha^2$.

$$\frac{X(x)}{X''(x)} = \frac{8T'(t)}{T(t)} = -\lambda$$

Separation is useful because we can break a partial differential equation into multiple ordinary differential equations.

Example

Generic form of the heat equation:

$$\frac{\partial u}{\partial t} = \beta \frac{\partial^2 u}{\partial x^2}$$

We want to find a solution that is nontrivial and bounded. It helps to narrow things down if we have boundary conditions and/or initial conditions.

$$u(0,t) = 0 \quad u(L,t) = 0 \quad u(x,0) = f(x)$$

We want to find $u(x,t)$ that satisfies the partial differential equation, its boundary conditions, and its initial condition. Since the equation is separable, we can substitute $u(x,t) = X(x)T(t)$ into the equation and show that

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{\beta T(t)}$$

If we choose the separation constant $-\lambda$:

$$\begin{aligned} \frac{X''(x)}{X(x)} &= -\lambda = \frac{T'(t)}{\beta T(t)} \\ X''(x) &= -\lambda X(x) \\ T'(t) &= -\lambda \beta T(t) \end{aligned}$$

With the boundary condition $u(0,t) = 0$, we know that $X(0)T(t) = 0$. With the boundary condition $u(L,t) = 0$, we know that $X(L)T(t) = 0$.

$$\begin{aligned} u(x,0) &= X(x)T(0) = f(x) \\ X(x) &= \frac{f(x)}{T(0)} \end{aligned}$$

We can start with either ordinary differential equation:

$$\begin{aligned} X''(x) &= -\lambda X(x) \quad X(0) = 0 \quad X(L) = 0 \\ X''(x) + \lambda X(x) &= 0 \\ r^2 + \lambda &= 0 \end{aligned}$$

Using this characteristic equation, we have three cases:

1. $\lambda < 0$

$$\begin{aligned}
r^2 &= -\lambda \\
r_{1,2} &= \pm\sqrt{-\lambda} \\
X(x) &= c_1 \mathrm{e}^{\sqrt{-\lambda}x} + c_2 \mathrm{e}^{-\sqrt{\lambda}x} \\
X(0) &= 0 = c_1 \mathrm{e}^0 + c_2 \mathrm{e}^0 \\
c_1 &= -c_2 \\
X(L) &= 0 = c_1(\mathrm{e}^{\sqrt{-\lambda}L} - \mathrm{e}^{-\sqrt{\lambda}L}) \\
0 &= c_1(\mathrm{e}^{2\sqrt{-\lambda}L} - 1) \\
c_1 &= 0 = c_2 \\
X(x) &= 0 \quad \text{(trivial solution)}
\end{aligned}$$

2. $\lambda = 0$

$$\begin{aligned}
X''(x) &= 0 \\
r^2 &= 0 \\
r &= 0 \\
X(x) &= c_1 \mathrm{e}^{0t} + c_2 x \mathrm{e}^{0t} \\
&= c_1 + x c_2 \\
X(0) &= 0 = c_1 \\
c_1 &= 0 \\
X(L) &= x c_2 \\
c_2 &= 0 \\
X(x) &= 0 \quad \text{(trivial solution)}
\end{aligned}$$

3. $\lambda > 0$

$$\begin{aligned}
r^2 + \lambda &= 0 \\
r_{1,2} &= \pm\sqrt{-\lambda} \\
\alpha^2 &= -\lambda \\
r_{1,2} &= \pm i\alpha = \pm i\sqrt{\lambda} \\
X(x) &= c_1 \cos(\alpha x) + c_2 \sin(\alpha x) \\
X(0) &= 0 = c_1 \cos(0) + c_2 \sin(0) \\
c_1 &= 0 \\
X(L) &= 0 = c_2 \sin(\alpha L)
\end{aligned}$$

If c_2 is 0, then the solution is again trivial. If it is not zero, then the solution depends on the separation constant.

$$\begin{aligned}
\sin(\alpha L) &= 0 \\
\alpha L &= n\pi, \quad n \in \mathbb{Z} \\
\alpha &= \frac{n\pi}{L} \\
X(x) &= c_2 \sin\left(\frac{n\pi}{L} x\right)
\end{aligned}$$

To produce a non-trivial solution for $X(x)$, we needed to use $\sqrt{\lambda} = \frac{n\pi}{L}$. We can use this to solve the second ordinary differential equation.

$$\begin{aligned}
T'(t) &= -\lambda\beta T(t) \\
T'(t) &= -\frac{n^2\pi^2}{L^2}\beta T(t) \\
T(t) &= c_2 \mathrm{e}^{ct} \\
T'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(T(t)) \\
&= c_2 c \mathrm{e}^{ct} \\
&= cT(t) \\
T(t) &= c_3 \mathrm{e}^{-\frac{n^2\pi^2}{L^2}\beta t}
\end{aligned}$$

We can substitute this back into $u(x,t) = X(x)T(t)$ to find our solution.

$$\begin{aligned} u(x,t) &= X(x)T(t) \\ &= c_2 \sin(\frac{n\pi x}{L}) c_3 \mathrm{e}^{-\frac{n^2\pi^2}{L^2}\beta t} \\ u_n(x,t) &= c_n \sin(\frac{n\pi x}{L}) \mathrm{e}^{-(\frac{n\pi}{L})^2 \beta t} \end{aligned}$$

This has infinitely many solutions because haven't applied our initial condition yet.

Eigenvalues and Eigenfunctions

Thus far, we have found special $X(x)$ and λ values that made the boundary value problem solutions non-trivial. These are known as eigenfunctions:

$$\begin{aligned} X_n(x) &= c_n \sin(\frac{n\pi x}{L}) = \text{eigenfunctions of the BVP} \\ \lambda_n &= (\frac{n\pi}{L})^2 = \text{eigenvalues of the BVP} \end{aligned}$$

We can associate eigenvalues and eigenfunctions with operators.

$$\begin{aligned} X'' + \lambda X &= 0 \\ X'' &= -\lambda X \\ LX &= -\lambda X \\ Ly &= D^2 y = y'' \\ LX &= D^2 X = X'' \end{aligned}$$

Eigenvalues of a linear operator L are values for which $Ly = \lambda y$ has non-trivial solutions for the given boundary conditions. Eigenfunctions are functions $y_n(x)$ corresponding to eigenvalues λ_n. Problems that require you to find the eigenvalues and eigenfunctions of boundary value problems require you to find all non-trivial y_n solutions.

Superposition Principle

If $u_1, u_2, \ldots, u_n$ are solutions to a linear homogeneous differential equation, then $u(x,t) = \sum_{i=1}^{\infty} c_i u_i(x,t)$ is also a solution where c_i are arbitrary constants. If we look at the solution we acquired before:

$$u_n(x,t) = c_n \sin(\frac{n\pi x}{L}) \mathrm{e}^{-(\frac{n\pi}{L})^2 \beta t}$$

We can rewrite this as:

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L})\mathrm{e}^{-(\frac{n\pi}{L})^2\beta t}$$

Summary

To recap, these are the steps to solve this type of boundary value problem:

1. Separate the variables.
$$u(x,t) = X(x)T(t)$$
Plug these back into the partial differential equation to get two ordinary differential equations related by a separation constant λ or α^2.

2. Use the boundary conditions to solve for $X(x)$. After separation, we have a second order differential equation X, which yields an auxiliary equation whose roots depend on the sign of λ. This yields a family of solutions $X_1(x), X_2(x), \ldots$ (eigenfunctions) corresponding to the λ values $\lambda_1, \lambda_2, \ldots$ (eigenvalues).

3. Find $T_n(t)$ for each λ_n.

4. Solutions should look like $u_n(x,t) = X_n(x)T_n(t)$.

5. Use superposition and Fourier series (if needed) to find the rest of the arbitrary constants c_n that fit the initial conditions.

Continuing the Previous Example

$$\begin{aligned} u_t &= \beta u_{xx} \\ u(0,t) &= u(L,t) = 0 \\ u(x,0) &= f(x) = 1 \end{aligned}$$

We can apply the initial condition starting from the solution we acquired earlier.

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L})\mathrm{e}^{-(\frac{n\pi}{L})^2\beta t} \\ u(x,0) &= \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L}) = 1 = f(x) \end{aligned}$$

This becomes a Fourier Sine Series problem with $L = 1$ and $b_n = c_n$.

$$\begin{aligned} c_n &= \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi x}{L})\ \mathrm{d}x \\ &= 2\int_0^1 1\sin(n\pi x)\ \mathrm{d}x \\ &= \frac{2}{n\pi}(1-\cos(n\pi)) \end{aligned}$$

We can substitute this back into the formal solution.

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} c_n \sin(\frac{n\pi x}{L})\mathrm{e}^{-(\frac{n\pi}{L})^2\beta t} \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1-\cos(n\pi))\sin(n\pi x)\mathrm{e}^{-\beta(n\pi)^2 t} \end{aligned}$$

This is a solution to the boundary value problem when $f(x) = 1$ and $L = 1$.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech