

Boundary Value Problems

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August 2018 - December 2018

Double Fourier Series

We will use Double Fourier Series to solve partial differential equations with multiple independent variables, such as heat transfer through a plate.

$$\begin{aligned} f(x, y) = & \frac{a_{00}}{4} + \frac{1}{2} \sum_{n=1}^{\infty} \left[a_{0n} \cos\left(\frac{n\pi y}{L}\right) + b_{0n} \sin\left(\frac{n\pi y}{L}\right) \right] \\ & + \frac{1}{2} \sum_{m=1}^{\infty} \left[a_{m0} \cos\left(\frac{m\pi x}{K}\right) + c_{m0} \sin\left(\frac{m\pi x}{K}\right) \right] \\ & + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[a_{mn} \cos\left(\frac{n\pi y}{L}\right) + b_{mn} \sin\left(\frac{n\pi y}{L}\right) \right] \cos\left(\frac{m\pi x}{K}\right) \\ & + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[c_{mn} \cos\left(\frac{n\pi y}{L}\right) + d_{mn} \sin\left(\frac{n\pi y}{L}\right) \right] \sin\left(\frac{m\pi x}{K}\right) \end{aligned}$$

The coefficients are:

$$\begin{aligned}a_{mn} &= \frac{1}{KL} \int_{-L}^L \int_{-K}^K f(x, y) \cos\left(\frac{m\pi x}{K}\right) \cos\left(\frac{n\pi y}{L}\right) dx dy \\b_{mn} &= \frac{1}{KL} \int_{-L}^L \int_{-K}^K f(x, y) \cos\left(\frac{m\pi x}{K}\right) \sin\left(\frac{n\pi y}{L}\right) dx dy \\c_{mn} &= \frac{1}{KL} \int_{-L}^L \int_{-K}^K f(x, y) \sin\left(\frac{m\pi x}{K}\right) \cos\left(\frac{n\pi y}{L}\right) dx dy \\d_{mn} &= \frac{1}{KL} \int_{-L}^L \int_{-K}^K f(x, y) \sin\left(\frac{m\pi x}{K}\right) \sin\left(\frac{n\pi y}{L}\right) dx dy\end{aligned}$$

A subproblem of this is finding a Fourier series for $f(x, y)$ over a domain. To reduce the amount of work, look for signs of symmetry:

1. $f(x, y) = f(-x, y)$, even in x
2. $-f(x, y) = f(-x, y)$, odd in x
3. $f(x, y) = f(x, -y)$, even in y
4. $-f(x, y) = f(x, -y)$, odd in y

If $f(x, y)$ has some symmetry, then some coefficients in the Double Fourier Series will be zero.

Example

Given the function $f(x, y) = xy$, evaluate the coefficients for the Double Fourier Series.

$$a_{mn} = \frac{1}{KL} \int_{-L}^L \int_{-K}^K f(x, y) \cos\left(\frac{m\pi x}{K}\right) \cos\left(\frac{n\pi y}{L}\right) dx dy$$

Since $f(x, y)$ is odd in x and $\cos\left(\frac{m\pi x}{K}\right)$ is even in x , the product is odd. Evaluating the integral over the symmetric interval for x results in 0.

$$b_{mn} = \frac{1}{KL} \int_{-L}^L \int_{-K}^K f(x, y) \cos\left(\frac{m\pi x}{K}\right) \sin\left(\frac{n\pi y}{L}\right) dx dy$$

For the same reason as a_{mn} , the product of the functions integrated for x over the symmetric interval is 0.

$$c_{mn} = \frac{1}{KL} \int_{-L}^L \int_{-K}^K f(x, y) \sin\left(\frac{m\pi x}{K}\right) \cos\left(\frac{n\pi y}{L}\right) dx dy$$

$f(x, y)$ is odd in y and $\cos\left(\frac{n\pi y}{L}\right)$ is even in y . Their product is odd so the integral in y over the symmetric interval is 0.

$$d_{mn} = \frac{1}{KL} \int_{-L}^L \int_{-K}^K f(x, y) \sin\left(\frac{m\pi x}{K}\right) \sin\left(\frac{n\pi y}{L}\right) dx dy$$

We cannot conclude this for d_{mn} because the function products are even in both x and y , so we would have to evaluate the integral here. All the other terms would drop out in the Double Fourier Series.

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech