Boundary Value Problems

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Fourier Series

Terminology:

- RH: right hand
- LH: left hand
- RH limit: $\lim_{x \to x_0^+} f(x) = f(x_0^+)$
- LH limit: $\lim_{x \to x_0^+} f(x) = f(x_0^-)$
- RH derivative: $\lim_{x \to x_0^+} f'(x) = f'_+(x_0)$
- LH derivative: $\lim_{x \to x_0^-} f'(x) = f'_-(x_0)$
- Continuous function: $f(x_0^-) = f(x_0^+) = f(x_0)$
- $f(x)$ is smooth at x_0 when $f'(x)$ is continuous at x_0

Many types of piecewise functions we care about are piecewise continouous and piecewise smooth. A function $f(x)$ is piecewise continuous if it has at most a finite number of jump or removable discontinuities. A function f is piecewise smooth if $f'(x)$ is piecewise continuous.

Example

$$f(x) = \begin{cases} x^2 & ,x \geq 0 \\ -1 & ,x < 0 \end{cases}$$

This is piecewise smooth because

$$f'(x) = \begin{cases} 2x & ,x \geq 0 \\ 0 & ,x < 0 \end{cases}$$

and this is piecewise continuous. However, the function

$$f(x) = \begin{cases} \frac{1}{x-1} & ,x > \frac{1}{2} \\ 2x & ,x < \frac{1}{2} \end{cases}$$

is neither piecewise smooth nor piecewise continuous due to the infinite discontinuity at $x = 1$.

Fourier Series

A Fourier series of $f(x)$ on $-L < x < L$ is associated with:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}))$$

Euler formulas for coefficients:

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^{L} f(x) \cos(\frac{n\pi x}{L}) \, \mathrm{d}x \\ &= \frac{1}{L}(f, \cos(\frac{n\pi x}{L})) \quad n = 0, 1, 2, \ldots \\ b_n &= \frac{1}{L} \int_{-L}^{L} f(x) \sin(\frac{n\pi x}{L}) \, \mathrm{d}x \\ &= \frac{1}{L}(f, \sin(\frac{n\pi x}{L})) \quad n = 1, 2, 3, \ldots \end{aligned}$$

a_n, b_n are components of $f(x)$ along sin and cos. As long as $f(x)$ is piecewise smooth on $[-L, L]$, the Fourier series of $f(x)$ will converge to $\frac{1}{2}[f(x_0^+ + f(x_0^-))]$ at a point $x_0 \in [-L, L]$. At a point x_0, the Fourier series will converge to $\frac{f(x_0^-)+f(x_0^+)}{2}$, so for

$$f(x) = \begin{cases} 0 & ,0 < x \leq L \\ 1 & ,-L \leq x < 0 \end{cases}$$

the Fourier series of f converges to $\frac{f(0^-)-f(0^+)}{2} = \frac{1+0}{2} = \frac{1}{2}$ at $x = 0$. In general, if f has a discontinuity (jump or removable) at x_0, the Fourier series converges to

$$\frac{1}{2}[f(x_0^+) + f(x_0^-)]$$

If f is continuous at x_0, the Fourier series converges to

$$\frac{1}{2}[f(x_0^+) + f(x_0^-)] = f(x_0)$$

Example

Find the convergence of the Fourier series of

$$f(x) = \begin{cases} 2 & ,0 < x < 1 \\ 0 & ,-1 < x < 0 \end{cases}$$

at the following points: $x = 0, x = 0.5, x = -\frac{3}{4}$

$$\frac{f(0^-) + f(0^+)}{2} = \frac{0+2}{2} = 1$$
$$\frac{f(0.5^-) + f(0.5^+)}{2} = \frac{2+2}{2} = 2$$
$$\frac{f(\frac{-3}{4}^-) + f(\frac{-3}{4}^+)}{2} = \frac{0+0}{2} = 0$$

Deriving the Euler formulas

Euler formulas:

1. $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\ \mathrm{d}x$
2. $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\ \mathrm{d}x$
3. $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})\ \mathrm{d}x$

These formulas arise from orthogonality. Recall that $\{1, \cos(\frac{n\pi x}{L}), \sin(\frac{n\pi x}{L})\}$ is an orthogonal set on $[-L, L]$. Suppose that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})$$

If we integrate both sides:

$$\begin{aligned}\int_{-L}^{L} f(x)\ \mathrm{d}x &= \int_{-L}^{L} \frac{a_0}{2}\ \mathrm{d}x + \sum_{n=1}^{\infty}\{a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})\} \\ &= \int_{-L}^{L} \frac{a_0}{2}\ \mathrm{d}x + 0 \\ &= a_0 L\end{aligned}$$

If we then divide by L:

$$\frac{1}{L}\int_{-L}^{L} f(x)\ \mathrm{d}x = a_0$$

This extends and generalizes for a_n and b_n as well.

Note

Sometimes, one of the terms evaluates down to a cosine. For example:

$$a_n = \frac{2}{n^2\pi^2}(cos(n\pi) - 1)$$

This term is 0 when n is even and nonzero when n is odd since $cos(n\pi)$ oscillates between -1 and 1. Because of this, we only care about a_{2n-1} and we can rewrite the Fourier series with this.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n-1}\cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L})$$

Example

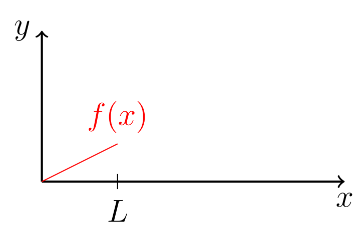
Find a_0 and $a_n, n \in \mathbb{N}$ for $f(x)$.

$$f(x) = \begin{cases} 1 & ,-2 < x < 0 \\ x & ,0 < x < 2 \end{cases}$$

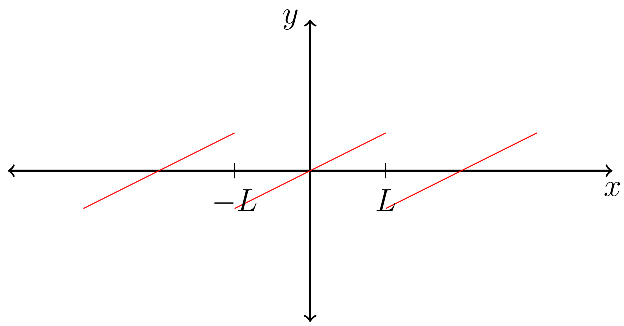
$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\ \mathrm{d}x \\
&= \frac{1}{2}\left\{\int_{-2}^{0} 1\ \mathrm{d}x + \int_{0}^{2} x\ \mathrm{d}x\right\} \\
&= \frac{1}{2}\left(\Big[x\Big]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2}\right) \\
&= \frac{1}{2}(2+2) \\
&= 2 \\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})\ \mathrm{d}x \\
&= \frac{1}{2}\left\{\int_{-2}^{0} 1\cos(\frac{n\pi x}{L})\ \mathrm{d}x + \int_{0}^{2} x\cos(\frac{n\pi x}{L})\ \mathrm{d}x\right\} \\
&= \frac{1}{2}\left\{\left[\frac{2}{n\pi}\sin(\frac{n\pi x}{2})\right]_{-2}^{0} + \left[x\frac{2}{n\pi}\sin(\frac{n\pi x}{2})\right]_{0}^{2} - \int_{0}^{2}\frac{2}{n\pi}\sin(\frac{n\pi x}{2})\ \mathrm{d}x\right\} \\
&= \frac{1}{2}\left\{0 + 0 + \frac{4}{n^2\pi^2}(\cos(n\pi) - 1)\right\} \\
&= \frac{2}{n^2\pi^2}(cos(n\pi) - 1) \\
&= \frac{2}{n^2\pi^2}(-1^n - 1)
\end{aligned}
$$

Fourier Sine and Cosine Functions

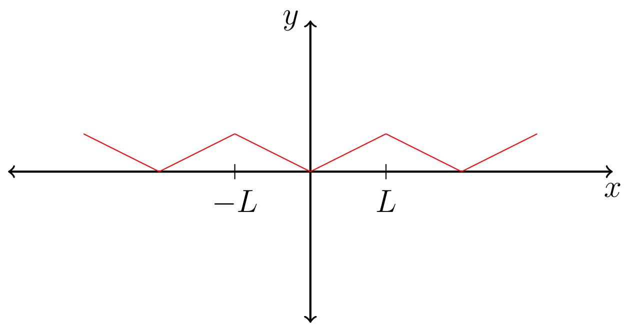
Suppose $f(x)$ is piecewise smooth on $0 < x < L$.



Because the period of $\cos(\frac{n\pi x}{L})$ and $\sin(\frac{n\pi x}{L})$ is $\frac{2L}{n}$, the Fourier series is 2L periodic. We can come up with special cases of the Fourier series by considering odd or even extensions of a function. We can define $f_o(x)$ as an odd 2L periodic extension of $f(x)$,



and $f_e(x)$ as an even 2L periodic extension of $f(x)$.



We define the Fourier Cosine Series (FCS) of $f(x)$ on $0 < x < L$ as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{n\pi x}{L})$$

$$a_0 = \frac{2}{L} \int_0^L f(x) \, \mathrm{d}x$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos(\frac{n\pi x}{L}) \, \mathrm{d}x$$

If $f(x)$ is piecewise smooth on $(0, L)$, the Fourier Cosine Series converges to

$$\frac{f_e(x_0^-) + f_e(x_0^+)}{2}$$

at $x = x_0$. The Fourier Sine Series (FSS) of $f(x)$ on $0 < x < L$ is defined as

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin(\frac{n\pi x}{L})$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L}) \, \mathrm{d}x$$

If $f(x)$ is piecewise smooth on $(0, L)$, the Fourier Sine Series converges to

$$\frac{f_o(x_1^-) + f_o(x_1^+)}{2}$$

at $x = x_1$. The Fourier Sine and Cosine Series are sometimes called half-range series.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech