

# Boundary Value Problems

Alvin Lin

August 2018 - December 2018

## Fourier Series

Terminology:

- RH: right hand
- LH: left hand
- RH limit:  $\lim_{x \rightarrow x_0^+} f(x) = f(x_0^+)$
- LH limit:  $\lim_{x \rightarrow x_0^-} f(x) = f(x_0^-)$
- RH derivative:  $\lim_{x \rightarrow x_0^+} f'(x) = f'_+(x_0)$
- LH derivative:  $\lim_{x \rightarrow x_0^-} f'(x) = f'_-(x_0)$
- Continuous function:  $f(x_0^-) = f(x_0^+) = f(x_0)$
- $f(x)$  is smooth at  $x_0$  when  $f'(x)$  is continuous at  $x_0$

Many types of piecewise functions we care about are piecewise continuous and piecewise smooth. A function  $f(x)$  is piecewise continuous if it has at most a finite number of jump or removable discontinuities. A function  $f$  is piecewise smooth if  $f'(x)$  is piecewise continuous.

### Example

$$f(x) = \begin{cases} x^2 & , x \geq 0 \\ -1 & , x < 0 \end{cases}$$

This is piecewise smooth because

$$f'(x) = \begin{cases} 2x & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

and this is piecewise continuous. However, the function

$$f(x) = \begin{cases} \frac{1}{x-1} & , x > \frac{1}{2} \\ 2x & , x < \frac{1}{2} \end{cases}$$

is neither piecewise smooth nor piecewise continuous due to the infinite discontinuity at  $x = 1$ .

### Fourier Series

A Fourier series of  $f(x)$  on  $-L < x < L$  is associated with:

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}))$$

Euler formulas for coefficients:

$$\begin{aligned} a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos(\frac{n\pi x}{L}) dx \\ &= \frac{1}{L} (f, \cos(\frac{n\pi x}{L})) \quad n = 0, 1, 2, \dots \\ b_n &= \frac{1}{L} \int_{-L}^L f(x) \sin(\frac{n\pi x}{L}) dx \\ &= \frac{1}{L} (f, \sin(\frac{n\pi x}{L})) \quad n = 1, 2, 3, \dots \end{aligned}$$

$a_n, b_n$  are components of  $f(x)$  along sin and cos. As long as  $f(x)$  is piecewise smooth on  $[-L, L]$ , the Fourier series of  $f(x)$  will converge to  $\frac{1}{2}[f(x_0^+) + f(x_0^-)]$  at a point  $x_0 \in [-L, L]$ . At a point  $x_0$ , the Fourier series will converge to  $\frac{f(x_0^-) + f(x_0^+)}{2}$ , so for

$$f(x) = \begin{cases} 0 & , 0 < x \leq L \\ 1 & , -L \leq x < 0 \end{cases}$$

the Fourier series of  $f$  converges to  $\frac{f(0^-)-f(0^+)}{2} = \frac{1+0}{2} = \frac{1}{2}$  at  $x = 0$ . In general, if  $f$  has a discontinuity (jump or removable) at  $x_0$ , the Fourier series converges to

$$\frac{1}{2}[f(x_0^+) + f(x_0^-)]$$

If  $f$  is continuous at  $x_0$ , the Fourier series converges to

$$\frac{1}{2}[f(x_0^+) + f(x_0^-)] = f(x_0)$$

### Example

Find the convergence of the Fourier series of

$$f(x) = \begin{cases} 2 & , 0 < x < 1 \\ 0 & , -1 < x < 0 \end{cases}$$

at the following points:  $x = 0, x = 0.5, x = -\frac{3}{4}$

$$\begin{aligned} \frac{f(0^-) + f(0^+)}{2} &= \frac{0 + 2}{2} = 1 \\ \frac{f(0.5^-) + f(0.5^+)}{2} &= \frac{2 + 2}{2} = 2 \\ \frac{f(\frac{-3}{4}^-) + f(\frac{-3}{4}^+)}{2} &= \frac{0 + 0}{2} = 0 \end{aligned}$$

### Deriving the Euler formulas

Euler formulas:

1.  $a_0 = \frac{1}{L} \int_{-L}^L f(x) \, dx$
2.  $a_n = \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx$
3.  $b_n = \frac{1}{L} \int_{-L}^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx$

These formulas arise from orthogonality. Recall that  $\{1, \cos(\frac{n\pi x}{L}), \sin(\frac{n\pi x}{L})\}$  is an orthogonal set on  $[-L, L]$ . Suppose that

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

If we integrate both sides:

$$\begin{aligned}\int_{-L}^L f(x) \, dx &= \int_{-L}^L \frac{a_0}{2} \, dx + \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right\} \\ &= \int_{-L}^L \frac{a_0}{2} \, dx + 0 \\ &= a_0 L\end{aligned}$$

If we then divide by  $L$ :

$$\frac{1}{L} \int_{-L}^L f(x) \, dx = a_0$$

This extends and generalizes for  $a_n$  and  $b_n$  as well.

### Note

Sometimes, one of the terms evaluates down to a cosine. For example:

$$a_n = \frac{2}{n^2 \pi^2} (\cos(n\pi) - 1)$$

This term is 0 when  $n$  is even and nonzero when  $n$  is odd since  $\cos(n\pi)$  oscillates between -1 and 1. Because of this, we only care about  $a_{2n-1}$  and we can rewrite the Fourier series with this.

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_{2n-1} \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

### Example

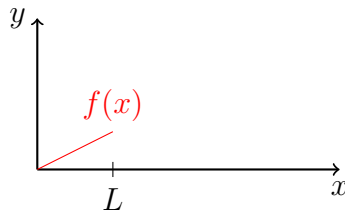
Find  $a_0$  and  $a_n, n \in \mathbb{N}$  for  $f(x)$ .

$$f(x) = \begin{cases} 1 & , -2 < x < 0 \\ x & , 0 < x < 2 \end{cases}$$

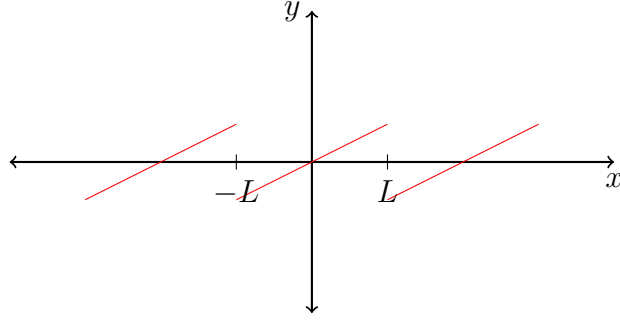
$$\begin{aligned}
a_0 &= \frac{1}{L} \int_{-L}^L f(x) \, dx \\
&= \frac{1}{2} \left\{ \int_{-2}^0 1 \, dx + \int_0^2 x \, dx \right\} \\
&= \frac{1}{2} \left( \left[ x \right]_{-2}^0 + \left[ \frac{x^2}{2} \right]_0^2 \right) \\
&= \frac{1}{2} (2 + 2) \\
&= 2 \\
a_n &= \frac{1}{L} \int_{-L}^L f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx \\
&= \frac{1}{2} \left\{ \int_{-2}^0 1 \cos\left(\frac{n\pi x}{L}\right) \, dx + \int_0^2 x \cos\left(\frac{n\pi x}{L}\right) \, dx \right\} \\
&= \frac{1}{2} \left\{ \left[ \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \right]_{-2}^0 + \left[ x \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \right]_0^2 - \int_0^2 \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \, dx \right\} \\
&= \frac{1}{2} \left\{ 0 + 0 + \frac{4}{n^2\pi^2} (\cos(n\pi) - 1) \right\} \\
&= \frac{2}{n^2\pi^2} (\cos(n\pi) - 1) \\
&= \frac{2}{n^2\pi^2} (-1^n - 1)
\end{aligned}$$

## Fourier Sine and Cosine Functions

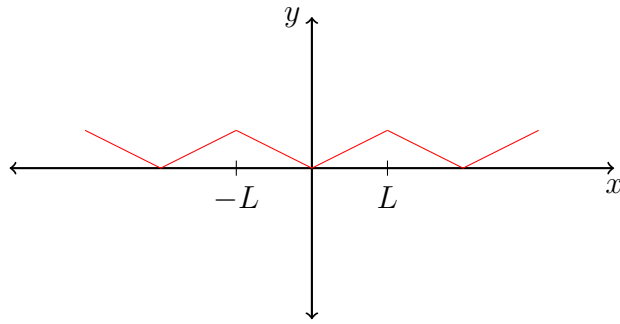
Suppose  $f(x)$  is piecewise smooth on  $0 < x < L$ .



Because the period of  $\cos\left(\frac{n\pi x}{L}\right)$  and  $\sin\left(\frac{n\pi x}{L}\right)$  is  $\frac{2L}{n}$ , the Fourier series is  $2L$  periodic. We can come up with special cases of the Fourier series by considering odd or even extensions of a function. We can define  $f_o(x)$  as an odd  $2L$  periodic extension of  $f(x)$ ,



and  $f_e(x)$  as an even  $2L$  periodic extension of  $f(x)$ .



We define the Fourier Cosine Series (FCS) of  $f(x)$  on  $0 < x < L$  as

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{2}{L} \int_0^L f(x) \, dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right) \, dx$$

If  $f(x)$  is piecewise smooth on  $(0, L)$ , the Fourier Cosine Series converges to

$$\frac{f_e(x_0^-) + f_e(x_0^+)}{2}$$

at  $x = x_0$ . The Fourier Sine Series (FSS) of  $f(x)$  on  $0 < x < L$  is defined as

$$f(x) \sim \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx$$

If  $f(x)$  is piecewise smooth on  $(0, L)$ , the Fourier Sine Series converges to

$$\frac{f_o(x_1^-) + f_o(x_1^+)}{2}$$

at  $x = x_1$ . The Fourier Sine and Cosine Series are sometimes called half-range series.

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)