Boundary Value Problems

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Vectors and Orthogonality

$$\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

or

$$\vec{v} = \begin{bmatrix} 2 & 4 \end{bmatrix}$$

Individual components of a vector are known as scalars, and are members of a field such as $\mathbb{R}$ or $\mathbb{C}$.

Adding vectors

$$\begin{aligned} \vec{a} &= \langle 1, 4 \rangle \\ \vec{b} &= \langle 3, 2 \rangle \\ \vec{a} + \vec{b} &= \langle 4, 6 \rangle \end{aligned}$$

Scalar Multiplication

$$\begin{aligned} 3\vec{a} &= 3\langle 1, 4 \rangle \\ &= \langle 3, 12 \rangle \end{aligned}$$

Magnitude

Given $\vec{a} = \langle a_1, a_2 \rangle$:

$$\|\vec{a}\| = \sqrt{(a_1)^2 + (a_2)^2}$$

This is also known as the length or norm of a vector. If $\vec{b} \in \mathbb{R}^n$:

$$\|\vec{b}\| = \sqrt{\sum_1^n (b_i)^2}$$

A **unit vector** is a vector of magnitude 1.

Linear Combinations

A **linear combination** of vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{v} = c_1\vec{a} + c_2\vec{b}$$

where each vector is multiplied by a constant c_i and added together.

Dot Product

The dot product is a special case of an inner product where:

$$\begin{aligned}
\vec{a} &= \langle a_1, a_2, \ldots, a_n \rangle \\
\vec{b} &= \langle b_1, b_2, \ldots, b_n \rangle \\
\vec{a}, \vec{b} &\in \mathbb{R} \\
\vec{a} \cdot \vec{b} &= (\vec{a}, \vec{b}) \\
&= a_1b_1 + a_2b_2 + \cdots + a_nb_n \\
&= \sum_1^n a_ib_i
\end{aligned}$$

The dot product is used to test for orthogonality. The dot product can also be calculated using the following property:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$$

where θ is the smaller of the two angles between $\vec{a}$ and $\vec{b}$.

Orthogonality

Any two vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if their dot product is zero. If a set of nonzero vectors $\{\vec{x_1}, \ldots, \vec{x_n}\}$ are all mutually orthogonal, that is

$$\vec{x_i} \cdot \vec{x_j} = 0 \text{ for all } i \neq j$$

then the set of vectors is an **orthogonal set**. An **orthonormal set** is an orthogonal set where all $\vec{x_i}$ are unit vectors.

$$\{\hat{\imath}, \hat{\jmath}, \hat{\jmath}\} \text{ is an orthonormal set}$$

$$\{3\hat{\imath}, \hat{\jmath}, \hat{k}\} \text{ is an orthogonal set but not orthonormal}$$

Orthogonal Functions

We can treat functions like vectors. For $f(x) = x^2$, as $x = 0, 1, 2, \ldots$, $f(x) = 0, 1, 4, \ldots$. This is an uncountably infinite list of numbers which we cannot write down. We can take linear combinations of functions (analogous to vectors). For example:

$$g(x) = 3\sin(x) - 2\cos(5x)$$

The function **norm** of $f : \mathbb{R} \to \mathbb{R}$ for f defined on $x \in [a, b]$ is defined as:

$$\|f(x)\| = \sqrt{\int_a^b (f(x))^2 \, \mathrm{d}x}$$

The **inner product** of two functions f and g is defined as:

$$(f, g) = (f(x), g(x)) = \int_a^b f(x)g(x) \, \mathrm{d}x$$

If $(f, g) = 0$, then f and g are orthogonal on the interval $[a, b]$ and vice versa. Note that $(f, f) = \|f\|^2$.

The set of functions $\{f_0(x), f_1(x), f_2(x), \ldots\}$ is an orthogonal set of functions if $(f_m, f_n) = 0$ whenever $m \neq n$. If $\{f_n\}$ is an orthogonal set where $\|f_n(x)\| = 1$ for $n = 0, 1, 2, \ldots$ then $\{f_n\}$ is an orthonormal set.

Example

Show that $\{\cos(\frac{n\pi x}{L})\}_{n=0}^{\infty} = \{1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \dots\}$ (where L is a given length greater than 0) is an orthogonal set on $[-L, L]$. First we need to check if

$$(f_n, f_m) = \int_{-L}^{L} \cos\frac{m\pi x}{L} \cos\frac{n\pi x}{L}\ \mathrm{d}x = 0 \text{ wherever } m \neq n$$

We can exploit the even and odd properties of functions to solve this. The first thing we should do is rewrite this function into a more familiar form using a trigonometric identity.

$$\cos(A)\cos(B) = \frac{1}{2}\Bigg[\cos(A-B) + \cos(A+B)\Bigg]$$

In the case where $m \neq n$:

$$\begin{aligned}
(f_n, f_m) &= \int_{-L}^{L} \cos(\frac{m\pi x}{L})\cos(\frac{n\pi x}{L})\ \mathrm{d}x \\
&= \frac{1}{2}\left[\cos\left(\frac{(m-n)\pi x}{L}\right) + \cos\left(\frac{(m+n)\pi x}{L}\right)\right]\ \mathrm{d}x \\
&= \frac{1}{2}\left[\frac{L}{(m-n)\pi}\sin\left(\frac{(m-n)\pi x}{L}\right)\right]_{-L}^{L} + \frac{1}{2}\left[\frac{L}{(m+n)\pi}\sin\left(\frac{(m+n)\pi x}{L}\right)\right]_{-L}^{L} \\
&= (\dots)(\sin((m-n)\pi) - \sin(-(m-n)\pi)) \\
&= (\dots)(\sin(n\pi)) \text{ where } n \in \mathbb{Z} \\
&= 0 \therefore \\
(f_n, f_m) &= 0 \text{ for } m \neq n
\end{aligned}$$

Therefore, $\{\cos(\frac{n\pi x}{L})\}_{n=0}^{\infty}$ is orthogonal.

$$(f_n, f_m) = \begin{cases} 0, & m \neq n \\ L, & m = n \neq 0 \\ 2L, & m = n = 0 \end{cases}$$

Since the inner product is 2L when $m = n$, the set is not orthonormal because there exists no L where $L = 2L = 1$. We can normalize the set by choosing choefficients $c_n f_n$ such that $\|c_n f_n\| = 1$. Since we have already found the function norm, we know the following:

$$\|f_n(x)\| = \sqrt{(f_n, f_n)} = \begin{cases} \sqrt{L}, & n > 0 \\ \sqrt{2L}, & n = 0 \end{cases}$$

Choosing $c_n = \frac{1}{\|f_n(x)\|}$ yields us:

$$\left\{ \frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}} \cos(\frac{\pi x}{L}), \frac{1}{\sqrt{L}} \cos(\frac{2\pi x}{L}), \ldots \right\}$$

which is an orthonormal set on $[-L, L]$.

Shortcuts

Recall that:

- odd function $\times$ odd function = even function
- even function $\times$ even function = even function
- even function $\times$ odd function = odd function

We can use this to quickly determine if two functions are orthogonal.

$$(\sin(x), \cos(x)) = \int_{-L}^{L} \sin(x)\cos(x)\ \mathrm{d}x$$

Because $\sin(x)$ is odd and $\cos(x)$ is even, their product is an odd function. Thus their integral is 0 over any symmetric interval. We can use this to quickly determine that they are orthogonal over $[-L, L]$.

Fourier Series

We will use the inner product and orthogonality to find the Fourier series corresponding to $f(x)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(\frac{n\pi x}{L}) + b_n \sin(\frac{n\pi x}{L}))$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech