

# Boundary Value Problems

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## Vectors and Orthogonality

$$\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

or

$$\vec{v} = [2 \ 4]$$

Individual components of a vector are known as scalars, and are members of a field such as  $\mathbb{R}$  or  $\mathbb{C}$ .

## Adding vectors

$$\vec{a} = \langle 1, 4 \rangle$$

$$\vec{b} = \langle 3, 2 \rangle$$

$$\vec{a} + \vec{b} = \langle 4, 6 \rangle$$

## Scalar Multiplication

$$3\vec{a} = 3\langle 1, 4 \rangle$$

$$= \langle 3, 12 \rangle$$

## Magnitude

Given  $\vec{a} = \langle a_1, a_2 \rangle$ :

$$\|\vec{a}\| = \sqrt{(a_1)^2 + (a_2)^2}$$

This is also known as the length or norm of a vector. If  $\vec{b} \in \mathbb{R}^n$ :

$$\|\vec{b}\| = \sqrt{\sum_1^n (b_i)^2}$$

A **unit vector** is a vector of magnitude 1.

## Linear Combinations

A **linear combination** of vectors  $\vec{a}$  and  $\vec{b}$  is

$$\vec{v} = c_1\vec{a} + c_2\vec{b}$$

where each vector is multiplied by a constant  $c_i$  and added together.

## Dot Product

The dot product is a special case of an inner product where:

$$\begin{aligned}\vec{a} &= \langle a_1, a_2, \dots, a_n \rangle \\ \vec{b} &= \langle b_1, b_2, \dots, b_n \rangle \\ \vec{a}, \vec{b} &\in \mathbb{R} \\ \vec{a} \cdot \vec{b} &= (\vec{a}, \vec{b}) \\ &= a_1b_1 + a_2b_2 + \dots + a_nb_n \\ &= \sum_1^n a_ib_i\end{aligned}$$

The dot product is used to test for orthogonality. The dot product can also be calculated using the following property:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos \theta$$

where  $\theta$  is the smaller of the two angles between  $\vec{a}$  and  $\vec{b}$ .

## Orthogonality

Any two vectors  $\vec{a}$  and  $\vec{b}$  are orthogonal if and only if their dot product is zero. If a set of nonzero vectors  $\{\vec{x}_1, \dots, \vec{x}_n\}$  are all mutually orthogonal, that is

$$\vec{x}_i \cdot \vec{x}_j = 0 \text{ for all } i \neq j$$

then the set of vectors is an **orthogonal set**. An **orthonormal set** is an orthogonal set where all  $\vec{x}_i$  are unit vectors.

$\{\hat{i}, \hat{j}, \hat{k}\}$  is an orthonormal set

$\{3\hat{i}, \hat{j}, \hat{k}\}$  is an orthogonal set but not orthonormal

## Orthogonal Functions

We can treat functions like vectors. For  $f(x) = x^2$ , as  $x = 0, 1, 2, \dots$ ,  $f(x) = 0, 1, 4, \dots$ . This is an uncountably infinite list of numbers which we cannot write down. We can take linear combinations of functions (analogous to vectors). For example:

$$g(x) = 3 \sin(x) - 2 \cos(5x)$$

The function **norm** of  $f : \mathbb{R} \rightarrow \mathbb{R}$  for  $f$  defined on  $x \in [a, b]$  is defined as:

$$\|f(x)\| = \sqrt{\int_a^b (f(x))^2 dx}$$

The **inner product** of two functions  $f$  and  $g$  is defined as:

$$(f, g) = (f(x), g(x)) = \int_a^b f(x)g(x) dx$$

If  $(f, g) = 0$ , then  $f$  and  $g$  are orthogonal on the interval  $[a, b]$  and vice versa. Note that  $(f, f) = \|f\|^2$ .

The set of functions  $\{f_0(x), f_1(x), f_2(x), \dots\}$  is an orthogonal set of functions if  $(f_m, f_n) = 0$  whenever  $m \neq n$ . If  $\{f_n\}$  is an orthogonal set where  $\|f_n(x)\| = 1$  for  $n = 0, 1, 2, \dots$  then  $\{f_n\}$  is an orthonormal set.

### Example

Show that  $\{\cos(\frac{n\pi x}{L})\}_{n=0}^{\infty} = \{1, \cos \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \dots\}$  (where  $L$  is a given length greater than 0) is an orthogonal set on  $[-L, L]$ . First we need to check if

$$(f_n, f_m) = \int_{-L}^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = 0 \text{ wherever } m \neq n$$

We can exploit the even and odd properties of functions to solve this. The first thing we should do is rewrite this function into a more familiar form using a trigonometric identity.

$$\cos(A) \cos(B) = \frac{1}{2} \left[ \cos(A - B) + \cos(A + B) \right]$$

In the case where  $m \neq n$ :

$$\begin{aligned} (f_n, f_m) &= \int_{-L}^L \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{2} \left[ \cos\left(\frac{(m-n)\pi x}{L}\right) + \cos\left(\frac{(m+n)\pi x}{L}\right) \right] dx \\ &= \frac{1}{2} \left[ \frac{L}{(m-n)\pi} \sin\left(\frac{(m-n)\pi x}{L}\right) \right]_{-L}^L + \frac{1}{2} \left[ \frac{L}{(m+n)\pi} \sin\left(\frac{(m+n)\pi x}{L}\right) \right]_{-L}^L \\ &= (\dots)(\sin((m-n)\pi) - \sin(-(m-n)\pi)) \\ &= (\dots)(\sin(n\pi)) \text{ where } n \in \mathbb{Z} \\ &= 0 \therefore \end{aligned}$$

$$(f_n, f_m) = 0 \text{ for } m \neq n$$

Therefore,  $\{\cos(\frac{n\pi x}{L})\}_{n=0}^{\infty}$  is orthogonal.

$$(f_n, f_m) = \begin{cases} 0, & m \neq n \\ L, & m = n \neq 0 \\ 2L, & m = n = 0 \end{cases}$$

Since the inner product is  $2L$  when  $m = n$ , the set is not orthonormal because there exists no  $L$  where  $L = 2L = 1$ . We can normalize the set by choosing coefficients  $c_n f_n$  such that  $\|c_n f_n\| = 1$ . Since we have already found the function norm, we know the following:

$$\|f_n(x)\| = \sqrt{(f_n, f_n)} = \begin{cases} \sqrt{L}, & n > 0 \\ \sqrt{2L}, & n = 0 \end{cases}$$

Choosing  $c_n = \frac{1}{\|f_n(x)\|}$  yields us:

$$\left\{ \frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}} \cos\left(\frac{\pi x}{L}\right), \frac{1}{\sqrt{L}} \cos\left(\frac{2\pi x}{L}\right), \dots \right\}$$

which is an orthonormal set on  $[-L, L]$ .

### Shortcuts

Recall that:

- odd function  $\times$  odd function = even function
- even function  $\times$  even function = even function
- even function  $\times$  odd function = odd function

We can use this to quickly determine if two functions are orthogonal.

$$(\sin(x), \cos(x)) = \int_{-L}^L \sin(x) \cos(x) dx$$

Because  $\sin(x)$  is odd and  $\cos(x)$  is even, their product is an odd function. Thus their integral is 0 over any symmetric interval. We can use this to quickly determine that they are orthogonal over  $[-L, L]$ .

### Fourier Series

We will use the inner product and orthogonality to find the Fourier series corresponding to  $f(x)$ .

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)