Boundary Value Problems

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Introduction and Classifications

Recall that ordinary differential equations (ODEs) were often focused on initial value problems. For example:

$$y'' + 3y' + 2y = 0 \quad y(x_0) = 3 \quad y'(x_0) = -1$$

Boundary value problems consist of a differential equation and boundary conditions given at two or more values of the independent variable. For example:

$$y'' + 16y = 0 \quad y(x_0) = y_0 \quad y(x_1) = y_1$$

It is implied here that $x_0 \neq x_1$. Boundary conditions can vary greatly:

$$\begin{aligned} y'(x_0) &= y_0 \quad y(x_1) = y_1 \\ y(x_0) &= y_0 \quad y'(x_1) = y_1 \\ y'(x_0) &= y_0 \quad y'(x_1) = y_1 \end{aligned}$$

Notationally, we may write $\frac{\mathrm{d}y}{\mathrm{d}x}$ as D_y where D is an operator applied to the variable y. Similarly, we may also abbreviate $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ as DDy or D^2y. For example, the following two differential equations are equivalent:

$$\begin{aligned} y'' + 3y' + 2y &= 0 \\ D^2y + 3Dy + 2y &= 0 \end{aligned}$$

Suppose we define the operator $L = D^2 + 3D + 2$ (short for "linear" operator), then $Ly = 0$. Linear operations on y include $y, y', y'', 3y, 3y', 3xy, 5xy'$, etc. Non-linear operations on y include $y^2, \sin(y), yy'$, etc. Note that L is a linear operator if and only if $L(c_1y_1 + c_2y_2) = c_1Ly_1 + c_2Ly_2$. For example:

- $y' + 2xy = 0$ is a linear differential equation.
- $y' + 3x\sin(y) = 0$ is a non-linear differential equation.

For an ordinary differential equation, the order of the equation is the highest order of derivative that appears in the equation. Partial differential equations may contain partial derivatives, and their order is also defined by the highest partial derivative it contains.

The generic form of a linear differential operator is as follows:

$$L = \alpha_n(x)D^n + \alpha_{n-1}D^{n-1} + \cdots + \alpha_1(x)D + \alpha_0(x)$$

where α can either be a constant or coefficient in terms of x. If $\alpha_0, \alpha_1, \ldots, \alpha_n$ are all constants, then the differential equation is a constant coefficient equation. If any α depends explicitly on x, then the differential equation is a variable coefficient equation.

If an equation has the form $Ly = 0$, then the differential equation is linear and homogeneous. If an equation has the form $L(y) = f(x) \neq 0$, then we call the equation non-homogeneous. These properties all influence the type of solution we will use as well as the ease of solving the differential equation.

linear	low order	homogeneous	const coefficient	ODE
non-linear	high order	non-homogeneous	variable coefficient	PDE

- $2\sin(x)y'' + y + 6 = 0$
  ODE
  linear
  second order
  variable coefficient
  non-homogeneous

- $uu_{xxx} + xu_t = 0$
  PDE
  non-linear
  third order

Standard Form of a Second Order Linear PDE

A PDE is linear and second order if it can be written in the form:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

with addition and subtraction of terms on both sides where u is the dependent variable, x, y are the independent variables, and A, B, C, D, E, F, G are constants or explicit functions of x or y.

If $G(x, y) \equiv 0)$, then the PDE is homogeneous, otherwise it is non-homogeneous. The PDE is:

- hyperbolic if $B^2 - 4AC > 0$
- parabolic if $B^2 - 4AC = 0$
- elliptic if $B^2 - 4AC < 0$

Example: Heat Equation

$u(x, t)$ represents the temperature along a rod at time t and position x.

$$u_t = \beta u_{xx}$$

where β is real, constant, and greater than 0. If we rewrite this according to our standard form:

$$\begin{aligned}
\beta u_{xx} + 0u_{xt} + 0u_{tt} + 0u_x - u_t + 0 &= 0 \\
\beta u_{xx} - u_t &= 0 \\
A &= \beta \quad B = 0 \quad C = 0 \\
B^2 - 4AC &= 0
\end{aligned}$$

Therefore, this equation is parabolic.

Solving PDEs

Throughout this course, we will build a procedure for solving PDEs:

- Use separation of variables if possible to break a PDE into two more ODEs.
- Solve each ODE subproblem.
- Assemble a general solution out of the ODE solutions.
- Solve for coefficients in the general solution using the boundary conditions or initial conditions if available.

Example

$$
\begin{aligned}
y'' + 16y = 0 \quad y(0) = 0 \quad y(\frac{\pi}{2}) &= 0 \\
y &= \mathrm{e}^{rt} \\
r^2 + 16 &= 0 \\
r &= 0 \pm 4i \quad \alpha = 0 \quad \beta = 4 \\
y &= \mathrm{e}^{\alpha t}(c_1 \cos(\beta t) + c_2 \sin(\beta t)) \\
&= c_1 \cos(4x) + c_2 \sin(4x)
\end{aligned}
$$

This is a general solution to the ODE. We can apply the boundary conditions to try and solve for the unknown coefficients.

$$
\begin{aligned}
y(0) &= 0 \\
&= c_1 \cos(0) + c_2 \sin(0) \\
c_1 &= 0 \\
y(\frac{\pi}{2}) &= 0 \\
&= c_1 \cos\left(\frac{4\pi}{2}\right) + c_2 \sin\left(\frac{4\pi}{2}\right) \\
&= c_2 \sin\left(\frac{4\pi}{2}\right) \\
&= 0
\end{aligned}
$$

We can determine a specific answer for c_1, but c_2 can be any real number.

$$
y = c_2 \sin(4x) \quad c_2 \in \mathbb{R}
$$

If we were to change the boundary conditions to $y(0) = 0$ and $y(\frac{\pi}{2}) = 1$, then the boundary value problem would have no solution. Boundary value problems can have no solution, exactly one solution, or an infinite number of solutions.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech