

# Probability and Statistics

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## Expected Values and Covariance

Let  $X$  and  $Y$  be discrete random variables with joint pmf  $p(x, y)$ . The expected value of  $h(X, Y)$  is:

$$\begin{aligned} E[h(X, Y)] &= \sum_x \sum_y h(x, y)p(x, y) \\ &= \sum_x \sum_y h(x, y)P(X = x \text{ and } Y = y) \\ \mu_x &= \sum_x xp_x(x) \\ \mu_y &= \sum_y xp_y(y) \\ &= \sum_y yP(Y = y) \end{aligned}$$

$p_x(x)$  and  $p_y(y)$  are marginal probability mass functions of  $X$  and  $Y$ , respectively. Recall the formal definitions:

$$\begin{aligned}
 p_x(x) &= \sum_y p(x, y) \\
 p_y(y) &= \sum_x p(x, y) \\
 (\sigma_x)^2 &= \sum_x (x - \mu_x)^2 p_x(x) \\
 &= \sum_x (x - \mu_x)^2 P(X = x) \\
 (\sigma_y)^2 &= \sum_y (y - \mu_y)^2 p_y(y) \\
 \sigma_x &= \sqrt{(\sigma_x)^2} \\
 \sigma_y &= \sqrt{(\sigma_y)^2}
 \end{aligned}$$

The **covariance** between the jointly distributed random variables  $X$  and  $Y$  is:

$$\begin{aligned}
 Cov(X, Y) &= E \left[ (x - \mu_x)(y - \mu_y) \right] \\
 &= \sum_x \sum_y (x - \mu_x)(y - \mu_y) p(x, y)
 \end{aligned}$$

Let  $X$  and  $Y$  be jointly distributed random variables.

$$Cov(X, Y) = E(XY) - \mu_x \mu_y$$

where:

$$E(XY) = \sum_x \sum_y xy p(x, y)$$

## Correlation Coefficient

The correlation coefficient of  $X$  and  $Y$ , denoted by  $Corr(X, Y)$ ,  $\rho_{x,y}$ , or just  $\rho$ , is:

$$\rho_{x,y} = \frac{Cov(X, Y)}{\sigma_x \sigma_y}$$

If the variables  $a$  and  $c$  are both positive or both negative, then:

$$Corr(aX + b, cY + d) = Corr(X, Y)$$

$$-1 \leq \text{Corr}(X, Y) \leq 1$$

If  $X$  and  $Y$  with joint pmf  $p(x, y)$  and joint pdf  $f(x, y)$  are independent, the  $\rho = 0$ . If  $\rho = 0$ , then the random variables  $X$  and  $Y$  may or may not be independent.  $\rho = \pm 1$  if and only if  $Y = aX + b$  for some numbers  $a$  and  $b$  with  $a \neq 0$ .

## Uses of the Correlation Coefficient

Let  $X$  be the random variable for the height of a randomly selected person at RIT and let  $Y$  be the random variable for their weight.  $\text{Corr}(X, Y)$  is not likely to be 1 or -1 because there is no strong correlation between height and weight.

Let  $X$  be the average temperature of a randomly selected city in degrees Centigrade, with  $Y$  being the average temperature expressed in degrees Fahrenheit. The correlation coefficient in this case is 1.

## Example

Randomly select a card from a file containing three cards:

- (Name, Readiness Assessment Score, Common Core Score)
- (James, 58, 81)
- (Mike, 73, 78)
- (Jane, 76, 85)

Let  $X$  be the random variable for the readiness assessment score and  $Y$  be the random variable for the Common Core score. Recall that a random variable is a function. Find the correlation coefficient of  $X$  and  $Y$ .

$x$	$y$	$p(x, y)$	$p_x(x)$	$p_y(y)$	$p_x(x)p_y(y)$
58	81	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{9}$
73	78	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{9}$
76	85	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{9}$

Are  $X$  and  $Y$  independent?

No,  $p(x, y) \neq p_x(x)p_y(y)$ .

$$\begin{aligned}\mu_x &= \sum_x xp_x(x) \\ &= 58 \cdot p_x(58) + 73 \cdot p_x(73) + 76 \cdot p_x(76) \\ &= 58 \cdot \frac{1}{3} + 73 \cdot \frac{1}{3} + 76 \cdot \frac{1}{3} = 69 \\ \mu_y &= \sum_y p_y(y) \\ &= 81 \cdot \frac{1}{3} + 78 \cdot \frac{1}{3} + 85 \cdot \frac{1}{3} = \frac{244}{3} \\ (\sigma_x)^2 &= \sum_x (x - \mu_x)^2 p_x(x) \\ &= (58 - 69)^2 \cdot \frac{1}{3} + (73 - 69)^2 \cdot \frac{1}{3} + (76 - 69)^2 \cdot \frac{1}{3} \\ &= \frac{1}{3} [11^2 + 4^2 + 7^2] \\ &= 62 \\ (\sigma_y)^2 &= \sum_y (y - \mu_y)^2 p_y(y) \\ &= (81 - \frac{244}{9})^2 \cdot \frac{1}{3} + (78 - \frac{244}{9})^2 \cdot \frac{1}{3} + (85 - \frac{244}{9})^2 \cdot \frac{1}{3} \\ &= \frac{1}{3} [(\frac{1}{3})^2 + (-\frac{10}{3})^2 + (\frac{11}{3})^2] \\ &= \frac{74}{9}\end{aligned}$$

$$\begin{aligned}
Cov(X, Y) &= E\left[(X - \mu_x)(Y - \mu_y)\right] \\
&= E(XY) - \mu_x\mu_y \\
&= \left[\sum_x \sum_y xy p(x, y)\right] - \mu_x\mu_y \\
&= 58 \cdot 81 \cdot \frac{1}{3} + 73 \cdot 78 \cdot \frac{1}{3} + 76 \cdot 85 \cdot \frac{1}{3} - 69 \cdot \left(81 + \frac{1}{3}\right) \\
&= 5.33 \\
Corr(X, Y) &= \frac{5.33}{\sqrt{62}\sqrt{8.22}} = 0.236
\end{aligned}$$

The positive correlation indicates that we can reasonable expect there is a correlation between the Readiness Assessment score and the Common Core score. The correlation is not perfectly linear however.

## Example

A surveyor wishes to lay out a square region with each side having length  $L$ . However, because of measurement error, he instead lays out a rectangle in which both the north-south sides are of length  $X$  and the east-west sides are both of length  $Y$ . Suppose  $X$  and  $Y$  are independent and that each is uniformly distributed on the interval  $[L - A, L + A]$  where  $0 < A < L$ . What is the expected area of the resulting rectangle? Find the joint pdf of  $X$  and  $Y$ .

$$f(x, y) = \begin{cases} h & , L - A \leq x \leq L + A \text{ and } L - A \leq y \leq L + A \\ 0 & , \text{otherwise} \end{cases}$$

How do we find the value of  $h$ ?

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy \\
&= \int_{L-A}^{L+A} \int_{L-A}^{L+A} h \, dx \, dy \\
&= h \int_{L-A}^{L+A} \int_{L-A}^{L+A} dx \, dy \\
&= h(2A)(2A) \\
h &= \frac{1}{4A^2}
\end{aligned}$$

The expected value of the area of the rectangle is  $E(XY)$ .

$$\begin{aligned} E(XY) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy p(x, y) dx dy \\ &= \int_{L-A}^{L+A} \int_{L-A}^{L+A} xyh dx dy \\ &= h \int_{L-A}^{L+A} y \left[ \int_{L-A}^{L+A} x dx \right] dy \\ &= h \left[ \int_{L-A}^{L+A} x dx \right] \left[ \int_{L-A}^{L+A} y dy \right] \\ &= h \left[ \int_{L-A}^{L+A} x dx \right]^2 \\ &= h \left[ \frac{1}{2} (x^2)_{x=L-A}^{x=L+A} \right] \\ &= h \frac{1}{2} \left[ (L+A)^2 - (L-A)^2 \right]^2 \\ &= h \frac{1}{2} \left[ ((L+A) + (L-A))((L+A) - (L-A)) \right]^2 \\ &= h \frac{1}{2} \left[ 2L \cdot 2A \right]^2 \\ &= L^2 \end{aligned}$$

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