Probability and Statistics

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Standard Normal Distribution

Standard Normal Distribution is the normal distribution within $\mu = 0$ and $\sigma = 1$. The pdf of a standard normal random variable Z is:

$$f(\delta; 0, \sigma) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{\delta^2}{2}}$$

The cdf of Z is:

$$\begin{aligned}\Phi(\delta) &= P(Z \leq \delta)\\ &= \int_{-\infty}^{\delta} f(y; 0, 1)\ \mathrm{d}y\\ &= \int_{-\infty}^{\delta} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\ \mathrm{d}y\end{aligned}$$

Example

Find $P(Z \leq 1.25)$, where Z is the standard normal random variable.

$$\begin{aligned}P(Z \leq 1.25) &= \int_{\infty}^{1.25} \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}}\ \mathrm{d}y\\ &= \Phi(1.25)\\ &\approx 0.8944\end{aligned}$$

Example

Find $\eta(0.99)$, 99th percentile, for the standard normal variable Z.

$$\begin{aligned} 0.99 &= \int_{-\infty}^{\eta(0.99)} f(\delta; 0, 1)\ \mathrm{d}\delta \\ &= \Phi(\eta(0.99)) \\ \eta(0.99) &= 2.33 \end{aligned}$$

The Z_α

Z_α denotes the value on the z axis for which the α of the area under the z curve lies to the right of Z_α. Example:

$$\begin{aligned} Z_{0.1} &= \eta(0.9) \\ 0.9 &= \Phi(Z_{0.1}) \\ &= \int_{-\infty}^{\eta(0.9)} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}}\ \mathrm{d}y \\ Z_{0.1} &= 1.28 \end{aligned}$$

Relationship between normal distribution and standard normal distribution

Let X be the normal random variable pdf $f(x;\mu,\sigma)$, and Z be the standard normal random variable (with pdf $f(\delta;0,1)$). Derivation:

$$\begin{aligned}
F(x;\mu,\sigma) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma}} \mathrm{e}^{-\frac{(y-\mu)^2}{2\sigma^2}}\ \mathrm{d}y \\
\Phi(\delta) &= \int_{-\infty}^{\delta} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-\frac{y^2}{2}}\ \mathrm{d}y \\
Z &= \frac{X-\mu}{\sigma} \\
P(a \le X \le b) &= P(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}) \\
&= P(Z \le \frac{b-\mu}{\sigma}) - P(Z \le \frac{a-\mu}{\sigma}) \\
&= \Phi(\frac{b-\mu}{\sigma}) - \Phi(\frac{a-\mu}{\sigma}) \\
P(X \le a) &= P(Z \le \frac{a-\mu}{\sigma}) \\
P(X \ge b) &= 1 - P(X \le b) \\
&= 1 - P(Z \le \frac{b-\mu}{\sigma}) \\
&= 1 - \Phi(\frac{b-\mu}{\sigma})
\end{aligned}$$

Example

The breakdown voltage of a randomly chosen diode of a particular type is known to be normally distributed. Find the probability that a diode's breakdown voltage is within 1 standard deviation of its mean value. Experiment: select a diode and measure its breakdown voltage. X: the measured breakdown voltage.

$$\mu = E(X)$$

$$\sigma = \sqrt{V(X)}$$

$$
\begin{aligned}
P(\mu-\sigma \le X \le \mu+\sigma) &= \int_{\mu-\sigma}^{\mu+\sigma} f(x;\mu,\sigma)\ \mathrm{d}x \\
&= \int_{\mu-\sigma}^{\mu+\sigma} \frac{1}{\sqrt{2\pi\sigma}} \mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}\ \mathrm{d}x \\
&= P(\frac{(\mu-\sigma)-\mu}{\sigma} \le Z \le \frac{(\mu+\sigma)-\mu}{\sigma}) \\
&= P(-1 \le Z \le 1) \\
&= \Phi(1) - \Phi(-1) \\
&= 0.8413 - 0.1587 \text{ (z-table lookup)}
\end{aligned}
$$

Relationship between two percentiles

(100p)th percentile for normal (μ, σ):

$$
= \mu + \left[\text{(100p)th percentile for standard normal}\right] \times \sigma
$$

Example

The temperature reading from a thermocouple placed in a constant-temperature medium is normally distributed with mean μ, the actual temperature of the medium, and standard deviation σ. What would the value of σ have to be to ensure that 95% of all readings are within $.1^\circ$ of μ?
Random variable X that has a normal distribution with $E(X) = \mu$ and standard

deviation $\sqrt{V(X)} = \sigma$.

$$
\begin{aligned}
0.95 &= P(\mu - 0.1 \leq X \leq \mu + 0.1) \\
&= P(\frac{(\mu - 0.1) - \mu}{\sigma} \leq Z \leq \frac{(\mu + 0.1) - \mu}{\sigma}) \\
&= P\left(-\frac{0.1}{\sigma} \leq Z \leq \frac{0.1}{\sigma}\right) \\
&= \Phi\left(\frac{0.1}{\sigma}\right) - \Phi\left(\frac{-0.1}{\sigma}\right) \\
&= \Phi\left(\frac{0.1}{\sigma}\right) - \left(1 - \Phi\left(\frac{0.1}{\sigma}\right)\right) \\
&= 2\Phi\left(\frac{0.1}{\sigma}\right) - 1 \\
\frac{0.95 + 1}{2} &= \Phi\left(\frac{0.1}{\sigma}\right) \\
\Phi(\frac{0.1}{\sigma}) &= 0.975 \\
\frac{0.1}{\sigma} &= 1.96 \text{ (From z-table reverse lookup)} \\
\sigma &= \frac{0.1}{1.96} \\
&= 0.05102
\end{aligned}
$$

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech