

Probability and Statistics

Alvin Lin

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Independence

Two events A and B are independent if $P(A|B) = P(A)$, and are dependent otherwise. For example, with the case of rolling a die twice, the probability of rolling a 2 the second time given that a 2 occurred in the first roll is equal to the probability of rolling a 2 the second time. The events of rolling a 2 in the first roll and rolling a 2 in the second roll are independent of each other.

$$P(a\ 2\ in\ the\ second\ roll|a\ 2\ in\ the\ first\ roll) = \frac{1}{6}$$

$$P(a\ 2\ in\ the\ second) = \frac{6}{36} = \frac{1}{6}$$

Example

A basket has 3 balls marked R, B, and G. Select a ball 3 times without replacement.

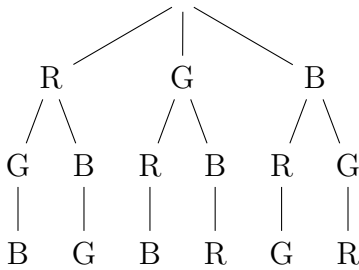
E_1 : a G in the first and a G in the second

E_2 : a G in the third

E_3 : a G at least once

E_4 : a G in the first

E_5 : a B in the first



$$P(A \cap B) = P(A|B)P(B) = P(A)P(B) \text{ (since } A \text{ and } B \text{ are independent)}$$

A and B are independent if and only if $P(A \cap B) = P(A)P(B)$.

$$P(E_1) = \frac{0}{6} = 0$$

$$P(E_1 \cap E_2) = 0$$

$$P(E_1 \cap E_2) = P(E_1)P(E_2)$$

E_1 and E_2 are independent.

$$P(E_4|E_5) = \frac{0}{2} = 0$$

$$P(E_4) = \frac{2}{6}$$

$$P(E_4|E_5) \neq P(E_4)$$

E_4 and E_5 are dependent.

$$P(E_4 \cap E_3) = \frac{2}{6}$$

$$P(E_4) = \frac{2}{6}$$

$$P(E_3) = \frac{6}{6}$$

$$P(E_4 \cap E_3) = P(E_4)P(E_3)$$

E_4 and E_3 are independent.

Mutual Independence of Multiple Events

Events A_1, A_2, \dots, A_n are mutually independent if for every $k(k = 2, 3, \dots, n)$ and for every subset of indices $i_1, i_2, i_3, \dots, i_k$ the following is true:

$$P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_k}) = P(A_{i_1}) \times P(A_{i_2}) \times \dots \times P(A_{i_k})$$

Example

Let A_1, A_2, A_3, A_4 be the events of an experiment such that A_1, A_2, A_3, A_4 are mutually independent.

$$\left[\begin{array}{l} P(A_1 \cap A_2) = P(A_1)P(A_2) \\ \wedge P(A_1 \cap A_3) = P(A_1)P(A_3) \\ \wedge P(A_1 \cap A_4) = P(A_1)P(A_4) \\ \dots \\ \wedge P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3) \\ \dots \\ \wedge P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(A_1)P(A_2)P(A_3)P(A_4) \end{array} \right]$$

Proposition

If $A_1, A_2, A_3, \dots, A_n$ are mutually independent, then $A'_1, A'_2, A'_3, \dots, A'_n$ are mutually independent where $A'_1 = \text{not } A_1$.

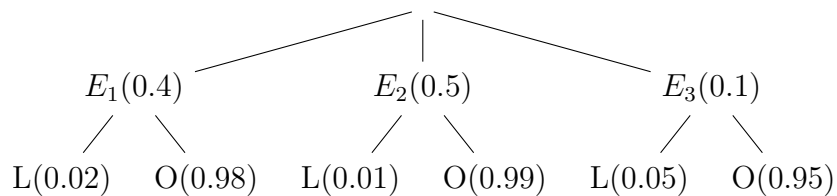
Consequently, if A_1, A_2, A_3, A_4 are mutually independent, then

$$P(\text{not } A_1 \cap \text{not } A_2 \cap \text{not } A_3) = P(\text{not } A_1)P(\text{not } A_2)P(\text{not } A_3)$$

Example

1. A certain company sends 40% of its overnight mail parcels via express mail service E_1 . Of these parcels, 2% arrive after the guaranteed delivery time (denote the event “late delivery” by L). If a record of an overnight mailing is randomly selected from the company’s file, what is the probability that the parcel went via E_1 and was late?

2. Suppose that 50% of the overnight parcels are sent via express mail service E_2 . Of those sent via E_2 , only 1% arrive late, whereas 5% of the parcels handled by E_3 arrive late. What is the probability that a randomly selected parcel arrives late?
3. If a randomly selected parcel arrived on time, what is the probability that it was not sent via E_1 ?



1.

$$(0.4)(0.02) = 0.008$$

2.

$$(0.4)(0.02) + (0.5)(0.01) + (0.1)(0.05) = 0.018$$

3.

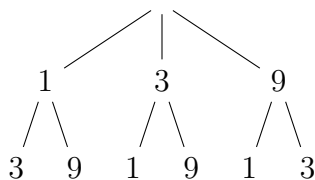
$$\frac{(0.5)(0.99) + (0.1)(0.95)}{(0.4)(0.98) + (0.5)(0.99) + (0.1)(0.95)}$$

Random Variable

For a given sample space S of some experiment, a random variable (rv) is any rule that associates a number with each outcome in S . A rv is a function whose domain is S and whose range is the set of real numbers.

Example

There are 3 balls marked 1, 3, 9 in a basket. Select a ball twice without replacement. Let X be the sum of the two numbers.



$$X(1, 3) = 4$$

$$X(3, 9) = 12$$

Any random variable whose only possible values are 0 and 1 is called a Bernoulli random variable.

Example

Determine the number of pumps in each of the two six-pump gas stations.

- X = the total number of pumps in use at the two stations
- Y = the difference between the number of pumps in use at station 1 and the number of pumps in use at station 2
- U = the maximum number of pumps in use at the two stations

$$W(\text{observed}) = (3, 4)$$

$$X(W) = 3 + 4 = 7$$

$$Y(W) = 3 - 4 = -1$$

$$U(W) = \max(3, 4) = 4$$

You can find all my notes at <http://omgimanerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech