Probability and Statistics

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Conditional Probability

Let A and B be events of an experiment. The conditional probability of A given that B has occurred is:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Idea Behind This Formula

Roll a die:

$$A : getting\ a\ 2$$
$$B : getting\ an\ even\ number$$

$$P(A) = \frac{\#\ of\ outcomes\ favorable\ to\ A}{total\ \#\ of\ outcomes}$$

(Under the assumption that the outcomes are equally likely)

Let's consider the probability of A given that B has occurred. Suppose the dice have detectors on the faces 2, 4, and 6. After we roll the dice, we can detect whether the dice has landed on a 2, 4, or 6, but we will not know which of the even numbers it has landed on.

$$P(A|B) = \frac{\#\ of\ outcomes\ favorable\ to\ A\ and\ in\ B}{total\ \#\ of\ outcomes\ in\ B}$$

$$P(A|B) = \frac{1}{3}$$

Considering the situation in the previous example, find $P(B|A)$.

$$P(B|A) = \frac{1}{1} = 1$$

If $P(B) > 0$, the conditional probability of A given that B has occurred is $P(A|B) = \frac{P(A\cap B)}{P(B)}$. Thus it follows that:

$$P(A \cap B) \times P(B) = P(A|B)$$

This holds true when $P(B) = 0$ as well.

Bayes' Theorem

Let $A_1, A_2, A_3, \ldots, A_k$ be events of an experiment. The events $A_1, A_2, A_3, \ldots, A_k$ are **mutually exclusive** or **disjoint** if and only if $A_1 \cap A_j = \varnothing$ for any $i, j \in \{1, 2, 3, \ldots, k\}$ with $i \neq j$. If $A_1, A_2, A_3, \ldots, A_k$ are mutually exclusive, then $A_1 \cap A_2 \cap A_3 = \varnothing$, $A_1 \cap A_2 \cap A_4 \cap A_6 = \varnothing$, and so on.

The events $A_1, A_2, A_3, \ldots, A_k$ are **exhaustive**, if and only if $A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_j = S$. If A_1 and A_2 are exhaustive, then:

$$\begin{aligned} P(A_1) + P(A_2) &= P(A_1 \cup A_2) + P(A_1 \cap A_2) \\ &(Inclusion\ Exclusion\ Principle) \\ &= P(S) + P(A_1 \cap A_2) \\ &(since\ A_1\ and\ A_2\ are\ exhaustive) \\ &= 1 + P(A_1 \cap A_2) \\ &\geq 1 \end{aligned}$$

Let's consider the contrapositive of the statement.

$$If\ P(A_1) + P(A_2) < 1,\ then\ A_1\ and\ A_2\ are\ not\ exhaustive.$$

The Law of Total Probability

Let $A_1, A_2, \ldots, A_k$ be mutually exclusive and exhaustive events. Then for any other event B:

$$\begin{aligned} P(B) &= P(B|A_1)P(A_1) + \cdots + P(B|A_k)P(A_j) \\ &= \sum_{i=1}^{k} P(B|A_i)P(A_i) \end{aligned}$$

Example

Roll a die:

$$\begin{aligned} B &: a\ 2\ or\ a\ 3 \\ A_1 &: a\ number\ less\ than\ 3 \\ A_2 &: a\ number\ greater\ than\ 2 \end{aligned}$$

	B	A_1	A_2	$P(B\|A_1)$	$P(B\|A_2)$
1		✓			
2	✓	✓		✓	
3	✓		✓		✓
4			✓		
5			✓		
6			✓		

$$\begin{aligned} P(B) &= \frac{2}{6} \\ P(B|A_1)P(A_1) + P(B|A_2)P(A_2) &= (\frac{1}{2})(\frac{2}{6}) + (\frac{1}{4})(\frac{4}{6}) \\ &= \frac{1}{6} + \frac{1}{6} \\ &= \frac{2}{6} \end{aligned}$$

Consistency Check:

$$A_1 \cap A_2 = \varnothing$$

$$A_1 \cup A_2 = S$$

A_1 and A_2 are mutually exclusive and exhaustive, so the law applies.

Example

An individual has 3 different email accounts. Most of her messages, in fact 70%, come into account #1, whereas 20% come into account #2, and the remaining 10% into account #3. Of the messages into account #1, only 1% are spam, whereas the corresponding percentages for accounts #2 and #3 are 2% and 5%, respectively. What is the probability that a randomly selected message is spam?

$$P(spam) = (0.7)(0.01) + (0.2)(0.02) + (0.1)(0.05)$$

Is this an application of the Law of Total Probability? Let:

$$\begin{aligned} B &: \textit{an email is spam} \\ A_1 &: \textit{an email goes to account \#1} \\ A_2 &: \textit{an email goes to account \#2} \\ A_3 &: \textit{an email goes to account \#3} \end{aligned}$$

$$A_1 \cap A_2 = \varnothing \quad A_1 \cap A_3 = \varnothing \quad A_2 \cap A_3 = \varnothing$$

The events A_1, A_2, A_3 are mutually exclusive.

$$A_1 \cup A_2 \cup A_3 = S$$

The events A_1, A_2, A_3 are exhaustive. So by the Law of Total Probability:

$$\begin{aligned} P(B) &= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3) \\ &= (0.01)(0.7) + (0.02)(0.2) + (0.05)(0.1) \end{aligned}$$

Another visualization

Fill a table with a convenient hypothetical total number of emails.

	account #1	account #2	account #3	total
spam	1×70	2×20	5×10	
not spam	99×70	98×20	95×10	
total	100×70	100×20	100×10	100×100

$$P(B) = \frac{(1)(70) + (2)(20) + (5)(10)}{(100)(100)}$$

We can also compute the conditional probabilities and check the consistency with the Law of Total Probability.

Bayes' Theorem (again)

Let $A_1, A_2, A_3, \ldots, A_k$ be a collection of k mutually exclusive and exhaustive events with prior probabilities $P(A_i)(i \in \{1, 2, \ldots, k\})$. Then for any other event B for which $P(B) > 0$, the posterior/conditional probability of A_j given that B has occurred is:

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=0}^{k} P(B|A_i)P(A_i)}$$

$$j \in \{1, 2, 3, \ldots, k\}$$

Derivation

$$\begin{aligned} P(A_j|B) = & \frac{P(A_j \cap B)}{P(B)} \\ & P(B) > 0 \\ & = \frac{P(B|A_j)P(A_j)}{P(B)} \\ & \textit{Multiplication Rule} \\ & = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{k} P(B|A_j)P(A_i)} \\ & \textit{The Law of Total Probability} \end{aligned}$$

Example

A recent Maryland highway safety study found that in 77% of all accidents the driver was wearing a seatbelt. Accident reports indicated that 92% of those drivers escaped serious injury (defined as hospitalization or death), but only 63% of the non-belted drivers were so fortunate. What is the probability that a driver who was seriously injured wasn't wearing a seatbelt?

Show the steps of finding the answer by defining $B_1, B_2 (= \textit{not } B_1), A_1, \textit{ and } A_2$ and then using Bayes' Theorem:

$$P(A_2|B_1) = \frac{P(B_1|A_2)P(A_2)}{\sum_{i=1}^{2} P(B_1|A_i)P(A_i)}$$

The probability that a driver who was seriously injured wasn't wearing a seatbelt is equal to the probability that a driver wasn't wearing a seatbelt given that he/she was seriously injured.

$$\begin{aligned} & B_1 : \textit{a driver was seriously injured} \\ & B_2 : \textit{a driver was not seriously injured} \\ & A_1 : \textit{a driver was wearing a seat belt} \\ & A_2 : \textit{a driver was not wearing a seat belt} \end{aligned}$$

The probability that we want to find is: $P(A_2|B_1)$. A_1 and A_2 are mutually exclusive and exhaustive.

$$
\begin{aligned}
P(A_1) &= 0.77 \\
P(A_2) &= 0.23 \\
P(B_2|A_1) &= 0.92 \\
P(B_1|A_1) &= 0.08 \\
P(B_2|A_2) &= 0.63 \\
P(B_1|A_2) &= 0.37 \\
P(A_2|B_1) &= \frac{P(B_1|A_2)P(A_2)}{P(B_1|A_1)P(A_1) + P(B_1A_2)P(A_2)} \\
&= \frac{0.37 \times 0.23}{0.08 \times 0.77 + 0.37 \times 0.23} \\
&\approx 0.58
\end{aligned}
$$

Show the steps of finding $P(A_2|B_1)$ using a tree diagram.

$$
\begin{array}{cccc}
\multicolumn{2}{c}{A_1(0.77)} & \multicolumn{2}{c}{A_2(0.23)} \\
B_1(0.08) & B_2(0.92) & B_1(0.37) & B_2(0.63)
\end{array}
$$

$$P(A_2|B_1) = \frac{0.23 \times 0.37}{0.77 \times 0.08 + 0.23 \times 0.37} \approx 0.58$$

Example

1 in 1000 adults is afflicted with a rare disease. Consider a test method. How exact is the test? If a person has the disease, there is a 99% chance they will test positive. If a person does not ahve the disease, there is a 2% chance they will test positive. If a randomly selected individual is tested and the result is positive, what is the probability that the individual has the disease?

By Bayes' Theorem, the probability is 4.7%.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech