# Linear Algebra

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## Final Exam Review

1. Solve a linear system
2. Like #8 from Exam 1. Given a 3 by 3 linear system with some coefficients $k$, find all values of $k$ such that the solution has no solutions, 1 solution, or infinitely many solution.
3. Similar to Exam 2, #3. A square matrix $M$ satisfies a given polynomial equation. Find $M^{-1}, rank(M), nullity(M)$.
4. Given a vector space $V$ and some subsets of $V$, determine if those subsets are subspaces of $V$.
5. Like Exam 2, #7. Given a 3 by 3 matrix $A$, with some entries in terms of $k$, find all values of $k$ making $A$ invertible.
6. Given 2 vector spaces $V$, determine if they are isomorphic.
7. Compute some determinants.
8. Similar to Exam 3 #5
9. Similar to Exam 3 #4
10. A question involving linear transformation $T : V \to W$ and the rank-nullity theorem.
11. Similar to questions from the Exam 4.
12. Similar to questions from the Exam 4.

**Bonus:** Given some vector space $V$, compute its dimension.
**Bonus:** Given transformation $T : V \to W$, find $[T]_{C \leftarrow B}$ where $B$ is a basis for $V$ and $C$ is a basis for $W$.

**Example**

Given a linear transformation $T : V \to W$ with $dim(V) = 5$ and $dim(W) = 6$. Are there any linear transformations $T : V \to W$ that are also onto? No, the rank-nullity theorem states that $dim(V) = rank(T) + nullity(T)$.

$$rank(T) = dim(range(T)) \leq dim(V) < dim(W)$$

$$range(T) \neq W$$

So $T$ is not onto.

Suppose $dim(V) = 6, dim(W) = 5$. Are there any one-to-one linear transformations $T : V \to W$?

$$\begin{aligned} rank(T) &\leq dim(W) = 5 \\ dim(V) &= rank(T) + nullity(T) \\ \therefore nullity(T) &\geq 1 \\ nullity(T) &\neq \{\vec{0}\} \end{aligned}$$

$T$ is not one-to-one.

**Example**

$V$ is a vector space with subspaces $U, W$. Prove $U \cap W$ is a subspace of $V$.

1. Is $\vec{0} \in U \cap W$? $U$ is a subspace so $\vec{0} \in U$. $W$ is a subspace so $\vec{0} \in W$. Therefore, $\vec{0} \in U \cap W$.
2. Let $\vec{x}, \vec{y} \in U \cap W$. So $\vec{x}, \vec{y} \in U$ and $\vec{x}, \vec{y} \in W$. Then $\vec{x} + \vec{y} \in U$ and $\vec{x} + \vec{y} \in W$. Thus, $\vec{x} + \vec{y} \in U \cap W$.
3. Let $\vec{u} \in U \cap W$ and $c$ be a scalar. $\vec{u} \in U$ and $\vec{u} \in W$. Then $c\vec{u} \in U$ and $c\vec{u} \in W$. Therefore, $c\vec{u} \in U \cap W$.

**Example**

Let $V$ be a vector space. Take $U, W$ as subspaces of $V$. Is it true that $U \cap W$ is a subspace of $V$? Let $V = \mathbb{R}^2$. $U = \{(x, 0) \mid x \in \mathbb{R}\}$ and $W = \{(0, y) \mid y \in \mathbb{R}\}$. $U \cap W$ is the union of the x and y axes, which is not closed under addition. For $U \cap W$ to be closed under addition, we need $U \subseteq W$ or $W \subseteq U$.

**Example**

Say $B$ is a set of vectors in a vector space $V$ with the property that every $\vec{v} \in V$ can be written uniquely as a linear combination of elements of $B$. Explain why $B$ is a basis for $V$. Show that the $span(B) = V$.

$$\text{Take } \vec{v} \in V$$
$$\vec{v} = \sum_{i=1}^{n} c_i \vec{v_i}$$
$$span(B) = V$$

Show that $B$ is linearly independent.

$$\sum_{i=1}^{n} c_i \vec{v_i} = \vec{0} = 0\vec{v_1} + 0\vec{v_2} + \cdots + 0\vec{v_n}$$

By uniqueness, $c_1 = c_2 = \cdots = c_n = 0$. Thus, $B$ is linearly independent, so $B$ is a basis for $V$.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech