Linear Algebra

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Review 2

Exam 2 covers section 2.3, 3.1, 3.2, 3.3, 3.5, 3.6, and homeworks 5, 6, and 7.

Example

Prove that, for square matrices A and B, $AB = BA$ if and only if $(A-B)(A+B) = A^2 - B^2$.

$$\begin{aligned}
\text{Assume } AB &= BA \\
(A-B)(A+B) &= A^2 + AB - BA - B^2 \\
&= A^2 - B^2
\end{aligned}$$

Converse:

$$\begin{aligned}
\text{Assume } (A-B)(A+B) &= A^2 - B^2 \\
A^2 - AB - BA - B^2 &= A^2 - B^2 \\
AB - BA &= 0 \\
AB &= BA
\end{aligned}$$

Example

Prove that the product of 2 upper triangular matrices is upper triangular.
Let A, B be upper triangular matrices. We want to show that $P = AB$ is upper

triangular:

$$\begin{aligned}
[P]_{ij} &= 0 \text{ iff } i > j \\
[P]_{ij} &= row_i(A) \cdot col_j(B) \\
&= \sum_{k=1}^{n} a_{ij} b_{kj} \\
&= \sum_{k=1}^{i-1} a_{ij} b_{kj} + \sum_{k=i}^{n} a_{ij} b_{kj} \\
&= \sum_{k=1}^{i-1} 0 b_{kj} + \sum_{k=i}^{n} a_{ij} 0 \\
&= 0
\end{aligned}$$

Example

Prove that the main diagonal of a skew-symmetric matrix must consist entirely of 0's.
A is **skew-symmetric** if $A^T = -A$.

$$[A]_{ii} = [A^T]_{ii} = [-A]_{ii}$$

For all i, $a_{ii} = -a_{ii}$, so $a_{ii} = 0$ for all i.

Example

Prove that is A and B are skew-symmetric, then so is $A + B$.

$$\begin{aligned}
(A+B)^T &= A^T + B^T \\
&= -A + (-B) \\
&= (-1)(A+B)
\end{aligned}$$

Therefore, $A + B$ is skew-symmetric.

Example

Prove that is A is an $n \times n$ matrix, then $A - A^T$ is skew symmetric.

$$\begin{aligned}
(A - A^T)^T &= A^T - (A^T)^T \\
&= A^T - A \\
&= (-1)(A - A^T)
\end{aligned}$$

Therefore $A - A^T$ is skew-symmetric.

Example

Definition: If A is a square matrix:

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

If A, B are $n \times n$ matrices, show that:

- $tr(A + B) = tr(A) + tr(B)$

$$\begin{aligned} tr(A + B) &= \sum_{i=1}^{n} (a_{ii} + b_{ii}) \\ &= \sum_{i=1}^{n} a_{ii} + \sum_{i=1}^{n} b_{ii} \\ &= tr(A) + tr(B) \end{aligned}$$

- $tr(kA) = k\ tr(A)$

$$\begin{aligned} tr(kA) &= \sum_{i=1}^{n} (ka_{ii}) \\ &= k \left(\sum_{i=1}^{n} a_{ii} \right) \\ &= k\ tr(A) \end{aligned}$$

Example

If A is any matrix, what does $tr(AA^T)$ equal.

$$\begin{aligned} tr(AA^T) &= \sum_{i=1}^{n} row_i(A) \cdot col_i(A^T) \\ &= \sum_{i=1}^{n} row_i(A) \cdot row_i(A) \\ &= \sum_{i=1}^{n} \|row_i(A)\|^2 \end{aligned}$$

Example

Show that if A is a square matrix satisfying $A^2 - 2A + I = 0$, then $A^{-1} = 2I - A$.

$$\begin{aligned} AA^{-1} &= I \\ A(2I - A) &= I \\ 2A - A^2 &= I \\ A^2 - 2A + I &= 0 \quad \text{given to be true} \\ A^{-1} = 2I - A & \end{aligned}$$

Example

Prove if a symmetric matrix is invertible, then its inverse is symmetric too. Let A be a symmetric and invertible matrix.

$$\begin{aligned} (A^{-1})^T &= (A^T)^{-1} \\ &= (A^{-1})^T \end{aligned}$$

Therefore, A^{-1} is symmetric.

Example

In $\mathbb{R}^2 : S = \{\begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0 \wedge y \geq 0\}$. Is S a subspace of $\mathbb{R}^2$?

Note that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in S$.

$$-1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin S$$

So S is not a subspace.

Example

Is $S = \{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid xy \geq 0\}$ a subspace of $\mathbb{R}^2$.

$$\begin{bmatrix} -50 \\ 0 \end{bmatrix} \in S$$

$$\begin{bmatrix} 1 \\ 100 \end{bmatrix} \in S$$

$$\begin{bmatrix} -50 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 100 \end{bmatrix} = \begin{bmatrix} -49 \\ 100 \end{bmatrix} \notin S$$

This subspace is not closed under addition and thus is not a subspace.

Example

Is $S = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x = 2x \wedge y = 0\}$ a subspace of $\mathbb{R}^3$? Why?

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$S = span(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix})$$

So S is a subspace.

Example

In $\mathbb{R}^2$, S consists of the union of the x-axis and y-axis. Is S a subspace of $\mathbb{R}^2$?

$$\vec{e_1} \in S \wedge \vec{e_2} \in S$$

$$\vec{e_1} + \vec{e_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin S$$

Therefore, S is not a subspace.

Example

A is a 3×5 matrix. Why are the column vectors of A linearly dependent.
Fact: If A is an $m \times n$ matrix:

$$\begin{aligned} rank(A) &\leq min(m, n) \\ A\vec{x} &= \vec{0} \\ 5 &= rank(A) + nullity(A) \\ rank(A) &\leq 3 \end{aligned}$$

The nullity of A must be at least 2, therefore $A\vec{x} = \vec{0}$ has a non-trivial solution. Thus, the columns of A are linearly dependent.

You can find all my notes at `http://omgimanerd.tech/notes`. If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech