

Linear Algebra

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Review 2

Exam 2 covers section 2.3, 3.1, 3.2, 3.3, 3.5, 3.6, and homeworks 5, 6, and 7.

Example

Prove that, for square matrices A and B , $AB = BA$ if and only if $(A - B)(A + B) = A^2 - B^2$.

$$\begin{aligned} \text{Assume } AB &= BA \\ (A - B)(A + B) &= A^2 + AB - BA - B^2 \\ &= A^2 - B^2 \end{aligned}$$

Converse:

$$\begin{aligned} \text{Assume } (A - B)(A + B) &= A^2 - B^2 \\ A^2 - AB - BA - B^2 &= A^2 - B^2 \\ AB - BA &= 0 \\ AB &= BA \end{aligned}$$

Example

Prove that the product of 2 upper triangular matrices is upper triangular.
Let A, B be upper triangular matrices. We want to show that $P = AB$ is upper

triangular:

$$\begin{aligned} [P]_{ij} &= 0 \text{ iff } i > j \\ [P]_{ij} &= \text{row}_i(A) \cdot \text{col}_j(B) \\ &= \sum_{k=1}^n a_{ik} b_{kj} \\ &= \sum_{k=1}^{i-1} a_{ik} b_{kj} + \sum_{k=i}^n a_{ik} b_{kj} \\ &= \sum_{k=1}^{i-1} 0 b_{kj} + \sum_{k=i}^n a_{ik} 0 \\ &= 0 \end{aligned}$$

Example

Prove that the main diagonal of a skew-symmetric matrix must consist entirely of 0's.

A is **skew-symmetric** if $A^T = -A$.

$$[A]_{ii} = [A^T]_{ii} = [-A]_{ii}$$

For all i , $a_{ii} = -a_{ii}$, so $a_{ii} = 0$ for all i .

Example

Prove that if A and B are skew-symmetric, then so is $A + B$.

$$\begin{aligned} (A + B)^T &= A^T + B^T \\ &= -A + (-B) \\ &= (-1)(A + B) \end{aligned}$$

Therefore, $A + B$ is skew-symmetric.

Example

Prove that if A is an $n \times n$ matrix, then $A - A^T$ is skew symmetric.

$$\begin{aligned} (A - A^T)^T &= A^T - (A^T)^T \\ &= A^T - A \\ &= (-1)(A - A^T) \end{aligned}$$

Therefore $A - A^T$ is skew-symmetric.

Example

Definition: If A is a square matrix:

$$\text{tr}(A) = \sum_{i=1}^n a_{ii}$$

If A, B are $n \times n$ matrices, show that:

- $\text{tr}(A + B) = \text{tr}(A) + \text{tr}(B)$

$$\begin{aligned}\text{tr}(A + B) &= \sum_{i=1}^n (a_{ii} + b_{ii}) \\ &= \sum_{i=1}^n a_{ii} + \sum_{i=1}^n b_{ii} \\ &= \text{tr}(A) + \text{tr}(B)\end{aligned}$$

- $\text{tr}(kA) = k \text{tr}(A)$

$$\begin{aligned}\text{tr}(kA) &= \sum_{i=1}^n (ka_{ii}) \\ &= k \left(\sum_{i=1}^n a_{ii} \right) \\ &= k \text{tr}(A)\end{aligned}$$

Example

If A is any matrix, what does $\text{tr}(AA^T)$ equal.

$$\begin{aligned}\text{tr}(AA^T) &= \sum_{i=1}^n \text{row}_i(A) \cdot \text{col}_i(A^T) \\ &= \sum_{i=1}^n \text{row}_i(A) \cdot \text{row}_i(A) \\ &= \sum_{i=1}^n \|\text{row}_i(A)\|^2\end{aligned}$$

Example

Show that if A is a square matrix satisfying $A^2 - 2A + I = 0$, then $A^{-1} = 2I - A$.

$$\begin{aligned}AA^{-1} &= I \\A(2I - A) &= I \\2A - A^2 &= I \\A^2 - 2A + I &= 0 \quad \text{given to be true} \\A^{-1} &= 2I - A\end{aligned}$$

Example

Prove if a symmetric matrix is invertible, then its inverse is symmetric too. Let A be a symmetric and invertible matrix.

$$\begin{aligned}(A^{-1})^T &= (A^T)^{-1} \\&= (A^{-1})^T\end{aligned}$$

Therefore, A^{-1} is symmetric.

Example

In \mathbb{R}^2 : $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \mid x \geq 0 \wedge y \geq 0 \right\}$. Is S a subspace of \mathbb{R}^2 ?

Note that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \in S$.

$$-1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \notin S$$

So S is not a subspace.

Example

Is $S = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid xy \geq 0 \right\}$ a subspace of \mathbb{R}^2 .

$$\begin{aligned} \begin{bmatrix} -50 \\ 0 \end{bmatrix} &\in S \\ \begin{bmatrix} 1 \\ 100 \end{bmatrix} &\in S \\ \begin{bmatrix} -50 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 100 \end{bmatrix} &= \begin{bmatrix} -49 \\ 100 \end{bmatrix} \notin S \end{aligned}$$

This subspace is not closed under addition and thus is not a subspace.

Example

Is $S = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid x = 2x \wedge y = 0 \right\}$ a subspace of \mathbb{R}^3 ? Why?

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$S = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}\right)$$

So S is a subspace.

Example

In \mathbb{R}^2 , S consists of the union of the x-axis and y-axis. Is S a subspace of \mathbb{R}^2 ?

$$\vec{e}_1 \in S \wedge \vec{e}_2 \in S$$

$$\vec{e}_1 + \vec{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin S$$

Therefore, S is not a subspace.

Example

A is a 3×5 matrix. Why are the column vectors of A linearly dependent.

Fact: If A is an $m \times n$ matrix:

$$\text{rank}(A) \leq \min(m, n)$$

$$A\vec{x} = \vec{0}$$

$$5 = \text{rank}(A) + \text{nullity}(A)$$

$$\text{rank}(A) \leq 3$$

The nullity of A must be at least 2, therefore $A\vec{x} = \vec{0}$ has a non-trivial solution. Thus, the columns of A are linearly dependent.

You can find all my notes at <http://omgimanagerd.tech/notes>. If you have any questions, comments, or concerns, please contact me at alvin@omgimanagerd.tech