

# Linear Algebra: Homework 8

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## Section 4.1

### Exercise 1

Show that  $\vec{v}$  is an eigenvector of  $A$  and find the corresponding eigenvalue.

$$\begin{aligned}A &= \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} \\ \vec{v} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ A\vec{v} &= \begin{bmatrix} 3 \\ 3 \end{bmatrix} \\ &= 3\vec{v} \\ \lambda &= 3\end{aligned}$$

### Exercise 3

Show that  $\vec{v}$  is an eigenvector of  $A$  and find the corresponding eigenvalue.

$$\begin{aligned}A &= \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix} \\ \vec{v} &= \begin{bmatrix} 1 \\ -2 \end{bmatrix} \\ A\vec{v} &= \begin{bmatrix} -3 \\ 6 \end{bmatrix} \\ &= -3\vec{v} \\ \lambda &= -3\end{aligned}$$

**Exercise 5**

Show that  $\vec{v}$  is an eigenvector of  $A$  and find the corresponding eigenvalue.

$$\begin{aligned}
 A &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -2 \\ 1 & 0 & 1 \end{bmatrix} \\
 \vec{v} &= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \\
 A\vec{v} &= \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix} \\
 &= 3\vec{v} \\
 \lambda &= 3
 \end{aligned}$$

**Exercise 7**

Show that  $\lambda$  is an eigenvalue of  $A$  and find one eigenvector corresponding to this eigenvalue.

$$\begin{aligned}
 A &= \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \\
 \lambda &= 3 \\
 A - \lambda I &= \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \\
 \left[ \begin{array}{cc|c} -1 & 2 & 0 \\ 2 & -4 & 0 \end{array} \right] &= \left[ \begin{array}{ccc} -1 & 2 & 0 \\ 0 & 0 & 0 \end{array} \right] \\
 x_1 &= 2x_2 \\
 \vec{x} &= \text{span} \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)
 \end{aligned}$$

One possible eigenvector is  $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ .

**Exercise 9**

Show that  $\lambda$  is an eigenvalue of  $A$  and find one eigenvector corresponding to this eigenvalue.

$$\begin{aligned}
 A &= \begin{bmatrix} 0 & 4 \\ -1 & 5 \end{bmatrix} \\
 \lambda &= 1 \\
 A - \lambda I &= \begin{bmatrix} -1 & 4 \\ -1 & 4 \end{bmatrix} \\
 \left[ \begin{array}{cc|c} -1 & 4 & 0 \\ -1 & 4 & 0 \end{array} \right] &= \left[ \begin{array}{ccc} -1 & 4 & 0 \\ 0 & 0 & 0 \end{array} \right] \\
 4x_2 &= x_1 \\
 \vec{x} &= \text{span} \left( \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right)
 \end{aligned}$$

One possible eigenvector is  $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ .

### Exercise 11

Show that  $\lambda$  is an eigenvalue of  $A$  and find one eigenvector corresponding to this eigenvalue.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 2 & 0 & 1 \end{bmatrix} \\ \lambda &= -1 \\ A - \lambda I &= \begin{bmatrix} 2 & 0 & 2 \\ -1 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix} \\ \left[ \begin{array}{ccc|c} 2 & 0 & 2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{array} \right] &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ x_1 &= -x_3 = x_2 \\ \vec{x} &= \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right) \end{aligned}$$

One possible eigenvector is  $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ .

### Exercise 27

Find all of the eigenvalues of the matrix  $A$  over the complex numbers  $\mathbb{C}$ . Give bases for each of the corresponding eigenspaces.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ |A - \lambda I| &= \begin{vmatrix} 1 - \lambda & 1 \\ -1 & 1 - \lambda \end{vmatrix} \\ &= (1 - \lambda)(1 - \lambda) + 1 \\ &= 1 - 2\lambda + \lambda^2 + 1 \\ &= \lambda^2 - 2\lambda + 2 = 0 \\ \lambda &= \frac{2 \pm \sqrt{4 - 8}}{2} \\ &= 1 \pm i \end{aligned}$$

$$A - (1 + i)I = \begin{bmatrix} -i & 1 \\ -1 & i \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array} \right] = \left[ \begin{array}{ccc} -i & 1 & 0 \\ -i & 1 & 0 \end{array} \right]$$

$$x_2 = ix_1$$

$$E_{1+i} = \text{span} \left( \begin{bmatrix} i \\ 1 \end{bmatrix} \right)$$

$$A - (1 - i)I = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} i & 1 & 0 \\ -1 & i & 0 \end{array} \right] = \left[ \begin{array}{ccc} i & 1 & 0 \\ -i & -1 & 0 \end{array} \right]$$

$$ix_1 = -1x_2$$

$$E_{1-i} = \text{span} \left( \begin{bmatrix} -i \\ 1 \end{bmatrix} \right)$$

### Exercise 29

Find all of the eigenvalues of the matrix  $A$  over the complex numbers  $\mathbb{C}$ . Give bases for each of the corresponding eigenspaces.

$$A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & i \\ i & 1 - \lambda \end{vmatrix}$$

$$= (1 - \lambda)^2 - (-1)$$

$$= 1 - 2\lambda + \lambda^2 + 1 = 0$$

$$\lambda = 1 \pm i$$

$$A - (1 + i)I = \begin{bmatrix} -i & i \\ i & -i \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} -i & i & 0 \\ i & -i & 0 \end{array} \right] = \left[ \begin{array}{ccc} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$x_1 = x_2$$

$$E_{1+i} = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

$$A - (1 - i)I = \begin{bmatrix} i & i \\ i & i \end{bmatrix}$$

$$\left[ \begin{array}{cc|c} i & i & 0 \\ i & i & 0 \end{array} \right] = \left[ \begin{array}{ccc} -1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$x_1 = -x_2$$

$$E_{1-i} = \text{span} \left( \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

## Section 4.2

### Exercise 1

Compute the determinant using cofactor expansion along the first row and column.

$$\begin{aligned}\begin{vmatrix} 1 & 0 & 3 \\ 5 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} &= 1 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 0 + 3 \begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} \\ &= 1(2 - 1) + 3(5) \\ &= 16 \\ &= 1 \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 5 \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} + 0 \\ &= 1(2 - 1) - 5(0 - 3) \\ &= 16\end{aligned}$$

### Exercise 3

Compute the determinant using cofactor expansion along the first row and column.

$$\begin{aligned}\begin{vmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} &= 1 \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} - (-1) \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} + 0 \\ &= 1(-1) + 1(1) \\ &= 0\end{aligned}$$

Since the matrix is symmetric, the expansion is the same along the first row and column.

### Exercise 5

Compute the determinant using cofactor expansion along the first row and column.

$$\begin{aligned}\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} &= 1 \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 3 & 1 \end{vmatrix} \\ &= 1(6 - 1) - 2(4 - 3) + 3(2 - 9) \\ &= 5 - 2 - 21 \\ &= -18\end{aligned}$$

Since the matrix is symmetric, the expansion is the same along the first row and column.

### Exercise 7

Compute the determinant using cofactor expansion along any row or column that seems convenient.

$$\begin{aligned}\begin{vmatrix} 5 & 2 & 2 \\ -1 & 1 & 2 \\ 3 & 0 & 0 \end{vmatrix} &= 3 \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} - 0 + 0 \\ &= 3(4 - 2) \\ &= 6\end{aligned}$$

**Exercise 9**

Compute the determinant using cofactor expansion along any row or column that seems convenient.

$$\begin{aligned} \begin{vmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{vmatrix} &= 1 \begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} - (-1) \begin{vmatrix} -4 & 3 \\ 2 & 4 \end{vmatrix} + 0 \\ &= 1(4 + 6) + 1(-16 - 6) \\ &= -12 \end{aligned}$$

**Exercise 11**

Compute the determinant using cofactor expansion along any row or column that seems convenient.

$$\begin{aligned} \begin{vmatrix} a & b & 0 \\ 0 & a & b \\ a & 0 & b \end{vmatrix} &= a \begin{vmatrix} a & b \\ 0 & b \end{vmatrix} - b \begin{vmatrix} 0 & b \\ a & b \end{vmatrix} + 0 \\ &= a(ab) - b(-ba) \\ &= a^2b + ab^2 \\ &= ab(a + b) \end{aligned}$$

**Exercise 13**

Compute the determinant using cofactor expansion along any row or column that seems convenient.

$$\begin{aligned} \begin{vmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{vmatrix} &= 0 - 1 \begin{vmatrix} 1 & 0 & 3 \\ 2 & 2 & 6 \\ 1 & 2 & 1 \end{vmatrix} + 0 - 0 \\ &= -1 \left( 1 \begin{vmatrix} 2 & 6 \\ 2 & 1 \end{vmatrix} - 0 + 3 \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} \right) \\ &= -1(1(2 - 12) + 3(4 - 2)) \\ &= 4 \end{aligned}$$

**Exercise 15**

Compute the determinant using cofactor expansion along any row or column that seems convenient.

$$\begin{aligned} \begin{vmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & c \\ 0 & d & e & f \\ g & h & i & j \end{vmatrix} &= 0 - 0 + 0 - a \begin{vmatrix} 0 & 0 & b \\ 0 & d & e \\ g & h & i \end{vmatrix} \\ &= -a(0 - 0 + b \begin{vmatrix} 0 & d \\ g & h \end{vmatrix}) \\ &= -ab(-dg) \\ &= abdg \end{aligned}$$

### Exercise 21

Prove Theorem 4.2: The determinant of a triangular matrix is the product of the entries on its main diagonal. Specifically, if  $A = [a_{ij}]$  is an  $n \times n$  triangular matrix, then

$$\det(A) = a_{11}a_{22} \cdots a_{nn}$$

Base Case:

$$|a_{11}| = a_{11}$$

Induction Hypothesis:

$$\det(A) = a_{11}a_{22}a_{33} \cdots a_{nn}$$

Induction:

$$\begin{aligned} \begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{(n+1)1} & a_{(n+1)2} & \cdots & a_{(n+1)(n+1)} \end{vmatrix} &= a_{11} \begin{vmatrix} a_{22} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ a_{(n+1)2} & \cdots & a_{(n+1)(n+1)} \end{vmatrix} \\ &= a_{11}a_{22}a_{33} \cdots a_{nn}a_{(n+1)(n+1)} \end{aligned}$$

### Exercise 23

Evaluate the given determinant using elementary row and/or column operations and Theorem 4.3 to reduce the matrix to row echelon form.

$$\begin{aligned} A &= \begin{bmatrix} -4 & 1 & 3 \\ 2 & -2 & 4 \\ 1 & -1 & 0 \end{bmatrix} \left( \frac{1}{2}R_2 \rightarrow R_2 \right) \\ &= \begin{bmatrix} -4 & 1 & 3 \\ 1 & -1 & 2 \\ 1 & -1 & 0 \end{bmatrix} \left( R_3 - R_2 \rightarrow R_3 \right) \quad (4R_2 \rightarrow R_2) \\ &= \begin{bmatrix} -4 & 1 & 3 \\ 4 & -4 & 8 \\ 0 & 0 & -2 \end{bmatrix} \left( R_2 + R_1 \rightarrow R_2 \right) \\ &= \begin{bmatrix} -4 & 1 & 3 \\ 0 & -3 & 11 \\ 0 & 0 & -2 \end{bmatrix} \\ \left( \frac{1}{2} \right) (4) \det(A) &= (-4)(-3)(-2) \\ \det(A) &= -12 \end{aligned}$$

**Exercise 25**

Evaluate the given determinant using elementary row and/or column operations and Theorem 4.3 to reduce the matrix to row echelon form.

$$\begin{aligned}
 A &= \begin{bmatrix} 2 & 0 & 3 & -1 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix} (R_1 - 2R_2 \rightarrow R_1) \\
 &= \begin{bmatrix} 0 & 0 & -1 & -5 \\ 1 & 0 & 2 & 2 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix} (R_4 - 2R_2 \rightarrow R_4) \\
 &= \begin{bmatrix} 0 & 0 & -1 & -5 \\ 0 & 0 & -3 & -7 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix} (R_2 - 3R_1 \rightarrow R_2) \\
 &= \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & -3 & -7 \\ 0 & -1 & 1 & 4 \\ 2 & 0 & 1 & -3 \end{bmatrix} \\
 (-2)(-3)\det(A) &= (2)(-1)(-3)(8) \\
 \det(A) &= 8
 \end{aligned}$$

**Exercise 27**

Use properties of determinants to evaluate the given determinant by inspection. Explain your reasoning.

$$\begin{vmatrix} 3 & 1 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 4 \end{vmatrix} = -24$$

The matrix is upper triangular. The determinant of a triangular matrix is the product of the elements on the diagonal.

**Exercise 29**

Use properties of determinants to evaluate the given determinant by inspection. Explain your reasoning.

$$\begin{vmatrix} 2 & 3 & -4 \\ 1 & -3 & -2 \\ -1 & 5 & 2 \end{vmatrix} = 0$$

Multiplying column 1 by -2 would make it identical to column 3.

**Exercise 31**

Use properties of determinants to evaluate the given determinant by inspection. Explain your reasoning.

$$\begin{vmatrix} 4 & 1 & 3 \\ -2 & 0 & -2 \\ 5 & 4 & 1 \end{vmatrix} = 0$$

Subtracting the second column from the first column makes the first column identical to the third column.



**Exercise 33**

Use properties of determinants to evaluate the given determinant by inspection. Explain your reasoning.

$$\begin{vmatrix} 0 & 2 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 \end{vmatrix} = -24$$

Swapping the first and second rows and swapping the third and fourth rows makes it a triangular matrix. These two operations negate the determinant and cancel out. The determinant of a triangular matrix is the product of the elements on the diagonal.

**Exercise 35**

Find the determinants assuming that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$$

$$\begin{vmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{vmatrix} = 8$$

**Exercise 37**

Find the determinants assuming that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$$

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -4$$

**Exercise 39**

Find the determinants assuming that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 4$$

$$\begin{vmatrix} 2c & b & a \\ 2f & e & d \\ 2i & h & g \end{vmatrix} = -8$$

**Exercise 45**

Use Theorem 4.6 to find all values of  $k$  for which  $A$  is invertible.

$$\begin{aligned}
 A &= \begin{bmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{bmatrix} \\
 0 &\neq \begin{vmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{vmatrix} \\
 &\neq k \begin{vmatrix} k+1 & 1 \\ -8 & k-1 \end{vmatrix} - 0 + k \begin{vmatrix} -k & 3 \\ k+1 & 1 \end{vmatrix} \\
 &\neq k(k^2 - 1 - (-8)) + k(-k - (3k + 3)) \\
 &\neq k^3 + 7k - k^2 - 3k^2 - 3k \\
 &\neq k^3 - 4k^2 + 4k \\
 &\neq k(k^2 - 4k + 4) \\
 &\neq k(k-2)(k-2) \\
 k &\neq 0 \quad k \neq 2
 \end{aligned}$$

**Exercise 46**

Use Theorem 4.6 to find all values of  $k$  for which  $A$  is invertible.

$$\begin{aligned}
 A &= \begin{bmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{bmatrix} \\
 0 &\neq \begin{vmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{vmatrix} \\
 &\neq k \begin{vmatrix} 2 & k \\ k & k \end{vmatrix} - k \begin{vmatrix} k^2 & k \\ 0 & k \end{vmatrix} + 0 \\
 &\neq k(2k - k^2) - k(k^3) \\
 &\neq 2k^2 - k^3 - k^4 \\
 &\neq -k^2(k^2 + k - 2) \\
 &\neq -k^2(k+2)(k-1) \\
 k &\neq 0 \quad k \neq -2 \quad k \neq 1
 \end{aligned}$$

**Exercise 47**

Assume that  $A$  and  $B$  are  $n \times n$  matrices with  $\det(A) = 3$  and  $\det(B) = -2$ . Find the indicated determinants.

$$\det(AB) = -6$$

**Exercise 48**

Assume that  $A$  and  $B$  are  $n \times n$  matrices with  $\det(A) = 3$  and  $\det(B) = -2$ . Find the indicated determinants.

$$\det(A^2) = 9$$

**Exercise 49**

Assume that  $A$  and  $B$  are  $n \times n$  matrices with  $\det(A) = 3$  and  $\det(B) = -2$ . Find the indicated determinants.

$$\det(B^{-1}A) = \frac{1}{-2}3 = -\frac{3}{2}$$

**Exercise 50**

Assume that  $A$  and  $B$  are  $n \times n$  matrices with  $\det(A) = 3$  and  $\det(B) = -2$ . Find the indicated determinants.

$$\det(2A) = 2^n(3)$$

**Exercise 51**

Assume that  $A$  and  $B$  are  $n \times n$  matrices with  $\det(A) = 3$  and  $\det(B) = -2$ . Find the indicated determinants.

$$\det(3B^T) = 3^n(-2)$$

**Exercise 52**

Assume that  $A$  and  $B$  are  $n \times n$  matrices with  $\det(A) = 3$  and  $\det(B) = -2$ . Find the indicated determinants.

$$\det(AA^T) = 9$$

**Exercise 57**

Use Cramer's Rule to solve the given linear system.

$$x + y = 1$$

$$x - y = 2$$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\det(A) = -2$$

$$\det(A_1\vec{b}) = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -3$$

$$\det(A_2\vec{b}) = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$$

$$x = \frac{-3}{-2} = \frac{3}{2}$$

$$y = \frac{1}{-2} = -\frac{1}{2}$$

**Exercise 58**

Use Cramer's Rule to solve the given linear system.

$$2x - y = 5$$

$$x + 3y = -1$$

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$\det(A) = 7$$

$$\det(A_1\vec{b}) = \begin{vmatrix} 5 & -1 \\ -1 & 3 \end{vmatrix} = 14$$

$$\det(A_2\vec{b}) = \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} = -7$$

$$x = \frac{14}{7} = 2$$

$$y = \frac{-7}{7} = -1$$

**Exercise 59**

Use Cramer's Rule to solve the given linear system.

$$2x + y + 3z = 1$$

$$y + z = 1$$

$$z = 1$$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\det(A) = 2$$

$$\det(A_1\vec{b}) = \begin{vmatrix} 1 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = -2$$

$$\det(A_2\vec{b}) = \begin{vmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$$\det(A_3\vec{b}) = \begin{vmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 2$$

$$x = \frac{-2}{2} = -1$$

$$y = \frac{0}{2} = 0$$

$$z = 1$$

### Exercise 60

Use Cramer's Rule to solve the given linear system.

$$x + y - z = 1$$

$$x + y + z = 2$$

$$x - y = 3$$

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\det(A) = 2 + 2 = 4$$

$$\det(A_1\vec{b}) = \begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{vmatrix} = 6 + 3 = 9$$

$$\det(A_2\vec{b}) = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 3 & 0 \end{vmatrix} = 3 - 6 = -3$$

$$\det(A_3\vec{b}) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & -1 & 3 \end{vmatrix} = 1 + 1 + 0 = 2$$

$$x = \frac{9}{4}$$

$$y = \frac{-3}{4}$$

$$z = \frac{2}{4}$$

### Exercise 61

Compute the inverse of the coefficient matrix.

$$A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$C_{11} = -1$$

$$C_{12} = -1$$

$$C_{21} = -1$$

$$C_{22} = 1$$

$$\begin{aligned} A^{-1} &= \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \end{aligned}$$

**Exercise 62**

Compute the inverse of the coefficient matrix.

$$\begin{aligned}
 A &= \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \\
 C_{11} &= 3 \\
 C_{12} &= -1 \\
 C_{21} &= 1 \\
 C_{22} &= 2 \\
 A^{-1} &= \frac{1}{7} \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix}^T \\
 &= \begin{bmatrix} \frac{3}{7} & \frac{1}{7} \\ -\frac{1}{7} & \frac{2}{7} \end{bmatrix}
 \end{aligned}$$

**Exercise 63**

Compute the inverse of the coefficient matrix.

$$\begin{aligned}
 A &= \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
 \text{adj}(A) &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -2 & -2 & 2 \end{bmatrix}^T \\
 \frac{1}{\det(A)} &= \frac{1}{2} \\
 A^{-1} &= \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

**Exercise 64**

Compute the inverse of the coefficient matrix.

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \\
 \text{adj}(A) &= \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & 2 \\ 2 & -2 & 0 \end{bmatrix}^T \\
 \frac{1}{\det(A)} &= \frac{1}{4} \\
 A^{-1} &= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}
 \end{aligned}$$

## Section 4.3

### Exercise 1

Compute the characteristic polynomial of  $A$ , the eigenvalues of  $A$ , a basis for each eigenspace of  $A$ , and the algebraic and geometric multiplicity of each eigenvalue.

$$\begin{aligned}A &= \begin{bmatrix} 1 & 3 \\ -2 & 6 \end{bmatrix} \\|A - \lambda I| &= \begin{vmatrix} 1 - \lambda & 3 \\ -2 & 6 - \lambda \end{vmatrix} = (6 - \lambda)(1 - \lambda) - (-6) \\&= 6 - 7\lambda + \lambda^2 + 6 = \lambda^2 - 7\lambda + 12 \\&= (\lambda - 3)(\lambda - 4) = 0 \\ \lambda &= 3 \quad \lambda = 4 \\[A - 3I|0] &= \begin{bmatrix} -2 & 3 & 0 \\ -2 & 3 & 0 \end{bmatrix} \\3x_2 &= 2x_1 \\E_3 &= \text{span} \left( \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) \\[A - 4I|0] &= \begin{bmatrix} -3 & 3 & 0 \\ -2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\x_1 &= x_2 \\E_4 &= \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)\end{aligned}$$

Each eigenvalue has algebraic and geometric multiplicity 1.

### Exercise 3

Compute the characteristic polynomial of  $A$ , the eigenvalues of  $A$ , a basis for each eigenspace of  $A$ , and the algebraic and geometric multiplicity of each eigenvalue.

$$\begin{aligned}A &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \\|A - \lambda I| &= \begin{vmatrix} 1 - \lambda & 1 & 0 \\ 0 & -2 - \lambda & 1 \\ 0 & 0 & 3 - \lambda \end{vmatrix} = (1 - \lambda)(-2 - \lambda)(3 - \lambda) = 0 \\ \lambda &= 1 \quad \lambda = -2 \quad \lambda = 3 \\[A - 1I|0] &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\x_2 &= x_3 = 0 \\E_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\end{aligned}$$

$$[A + 2I|0] = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$3x_1 = -x_2$$

$$x_3 = 0$$

$$E_{-2} = \text{span} \left( \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \right)$$

$$[A - 3I|0] = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 & -1 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$10x_1 = 5x_2 = x_3$$

$$E_3 = \text{span} \left( \begin{bmatrix} 1 \\ 2 \\ 10 \end{bmatrix} \right)$$

The algebraic and geometric multiplicities of the eigenvalues are 1.

### Exercise 5

Compute the characteristic polynomial of  $A$ , the eigenvalues of  $A$ , a basis for each eigenspace of  $A$ , and the algebraic and geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 1 - \lambda & 2 & 0 \\ -1 & -1 - \lambda & 1 \\ 0 & 1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)((1 - \lambda)(-1 - \lambda) - 1) - 2(-(1 - \lambda)) \\ &= (1 - \lambda)(-1 + \lambda^2 - 1) + 2 - 2\lambda \\ &= -2 + \lambda^2 + 2\lambda - \lambda^3 + 2 - 2\lambda \\ &= -\lambda^3 + \lambda^2 \\ &= -\lambda^2(\lambda - 1) = 0 \end{aligned}$$

$$\lambda = 0 \quad \lambda = 1$$

$$[A - 0I|0] = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 2x_3$$

$$x_2 = -x_3$$

$$E_0 = \text{span} \left( \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right)$$



$$[A - 1I|0] = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_3$$

$$x_2 = 0$$

$$E_1 = \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

$\lambda = 0$  has algebraic multiplicity 2 while  $\lambda = 1$  has algebraic multiplicity 1. They both have geometric multiplicity 1.

### Exercise 7

Compute the characteristic polynomial of  $A$ , the eigenvalues of  $A$ , a basis for each eigenspace of  $A$ , and the algebraic and geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ -1 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 4 - \lambda & 0 & 1 \\ 2 & 3 - \lambda & 2 \\ -1 & 0 & 2 - \lambda \end{vmatrix} = (4 - \lambda)((3 - \lambda)(2 - \lambda)) + (0 + (3 - \lambda)) \\ &= (4 - \lambda)(6 - 5\lambda + \lambda^2) + 3 - \lambda \\ &= 24 - 20\lambda + 4\lambda^2 - 6\lambda + 5\lambda^2 - \lambda^3 + 3 - \lambda \\ &= -\lambda^3 + 9\lambda^2 - 27\lambda + 27 = 0 \end{aligned}$$

$$\lambda = 3$$

$$[A - 3I|0] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ -1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_3$$

$$E_3 = [x_1 \quad x_2 \quad -x_1]$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

$\lambda = 3$  has algebraic multiplicity 3 and geometric multiplicity 2.

### Exercise 9

Compute the characteristic polynomial of  $A$ , the eigenvalues of  $A$ , a basis for each eigenspace of  $A$ , and the algebraic and geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 3 - \lambda & 1 & 0 & 0 \\ -1 & 1 - \lambda & 0 & 0 \\ 0 & 0 & 1 - \lambda & 4 \\ 0 & 0 & 1 & 1 - \lambda \end{vmatrix} \\ &= ((3 - \lambda)(1 - \lambda) + 1)((1 - \lambda)^2 - 4) \\ &= \lambda^4 - 6\lambda^3 + 9\lambda^2 + 4\lambda - 12 \\ &= \lambda^2(\lambda^2 - 6\lambda + 9) + 4(\lambda - 3) \\ &= (\lambda - 3)(\lambda^3 - 3\lambda^2 + 4) \\ &= (\lambda - 3)(\lambda - 2)^2(\lambda + 1) = 0 \\ \lambda &= -1 \quad \lambda = 2 \quad \lambda = 3 \end{aligned}$$

$$[A + 1I|0] = \begin{bmatrix} 4 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_2 = 0$$

$$x_3 = -2x_4$$

$$E_{-1} = \text{span} \left( \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right)$$

$$[A - 2I|0] = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_2$$

$$x_3 = x_4 = 0$$

$$E_2 = \text{span} \left( \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$[A - 3I|0] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & 4 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = x_2 = 0$$

$$x_3 = 2x_4$$

$$E_3 = \text{span} \left( \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right)$$

### Exercise 11

Compute the characteristic polynomial of  $A$ , the eigenvalues of  $A$ , a basis for each eigenspace of  $A$ , and the algebraic and geometric multiplicity of each eigenvalue.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 \\ -2 & 1 & 2 & -1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & 0 & 0 & 0 \\ 0 & 1 - \lambda & 0 & 0 \\ 1 & 1 & 3 - \lambda & 0 \\ -2 & 1 & 2 & -1 - \lambda \end{vmatrix}$$

$$= (1 - \lambda)(1 - \lambda)(3 - \lambda)(-1 - \lambda) = 0$$

$$\lambda = 1 \quad \lambda = 3 \quad \lambda = -1$$

$$[A - 1I|0] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ -2 & 1 & 2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 3 & 0 & 0 & 2 & 0 \end{bmatrix}$$

$$x_1 + x_2 + 2x_3 = 0$$

$$3x_1 + 2x_4 = 0$$

$$E_1 = \begin{bmatrix} x_1 \\ -2x_3 - x_1 \\ x_3 \\ \frac{-3x_1}{2} \end{bmatrix} = \begin{bmatrix} 2x_1 \\ -4x_3 - 2x_1 \\ 2x_3 \\ -3x_1 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ -2 \\ 0 \\ -3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -4 \\ 2 \\ 0 \end{bmatrix}$$

$$= \text{span} \left( \begin{pmatrix} 2 \\ -2 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ -4 \\ 2 \\ 0 \end{pmatrix} \right)$$

### Exercise 13

Prove Theorem 4.18b: Let  $A$  be a square matrix with eigenvalue  $\lambda$  and corresponding eigenvector  $\vec{x}$ . If  $A$  is invertible, then  $\frac{1}{\lambda}$  is an eigenvalue of  $A^{-1}$  with corresponding eigenvector  $\vec{x}$ .

$$\begin{aligned}A\vec{x} &= \lambda\vec{x} \\ A^{-1}A\vec{x} &= A^{-1}(\lambda\vec{x}) \\ \vec{x} &= \lambda A^{-1}\vec{x} \\ A^{-1}\vec{x} &= \frac{1}{\lambda}\vec{x}\end{aligned}$$

### Exercise 14

Prove Theorem 4.18c: Let  $A$  be a square matrix with eigenvalue  $\lambda$  and corresponding eigenvector  $\vec{x}$ . If  $A$  is invertible, then for any integer  $n$ ,  $\lambda^n$  is an eigenvalue of  $A^n$  with corresponding eigenvector  $\vec{x}$ .

Base Case  $n = 1$ :

$$A^1\vec{x} = \lambda^1\vec{x}$$

Induction Hypothesis:  $A^n\vec{x} = \lambda^n\vec{x}$

Induction Step:

$$\begin{aligned}A^{n+1}\vec{x} &= A(A^n\vec{x}) \\ &= A(\lambda^n\vec{x}) \\ &= \lambda^n(A\vec{x}) \\ &= \lambda^n(\lambda\vec{x}) \\ &= \lambda^{n+1}\vec{x}\end{aligned}$$

### Exercise 20

Let  $A$  be a nilpotent matrix. Show that  $\lambda = 0$  is the only eigenvalue of  $A$ .

$$|A^m| = |0| = 0$$

Thus  $\lambda = 0$  is an eigenvalue. Show that there are no other eigenvalues.

Suppose  $\lambda$  is another eigenvalue ( $\lambda \neq 0$ ). Then  $\lambda^m$  is an eigenvalue of  $A^m$ . This forces  $\lambda^m = 0 \therefore \lambda = 0$ .

### Exercise 21

Let  $A$  be an idempotent matrix. Show that  $\lambda = 0$  and  $\lambda = 1$  are the only possible eigenvalues of  $A$ .

Since  $A$  is idempotent, it means that  $A^2 = A$ . Let  $\lambda$  be an eigenvalue of  $A$ , then  $\lambda^2$  is an eigenvalue of  $A^2 = A$ .

$$\begin{aligned}\lambda^2 &= \lambda \\ \lambda^2 - \lambda &= 0 \\ \lambda(\lambda - 1) &= 0 \\ \lambda = 0 \quad \lambda = 1\end{aligned}$$

### Exercise 22

If  $\vec{v}$  is an eigenvector of  $A$  with corresponding eigenvalue  $\lambda$  and  $c$  is a scalar, show that  $\vec{v}$  is an eigenvector of  $A - cI$  with corresponding eigenvalue  $\lambda - c$ .

$$\begin{aligned} B &= A - cI \\ (A - \lambda I)\vec{v} &= 0 \\ B - (\lambda - c)I &= A - cI - (\lambda - c)I \\ &= A - cI - \lambda I + cI \\ &= A - \lambda I \\ (B - (\lambda - c)I)\vec{v} &= (A - \lambda I)\vec{v} = 0 \end{aligned}$$

## Section 4.4

### Exercise 1

Show that  $A$  and  $B$  are not similar matrices.

$$\begin{aligned} A &= \begin{bmatrix} 4 & 1 \\ 3 & 1 \end{bmatrix} & B &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ |A - \lambda I| &= \begin{vmatrix} 4 - \lambda & 1 \\ 3 & 1 - \lambda \end{vmatrix} = (4 - \lambda)(1 - \lambda) - 3 \\ &= 4 - 5\lambda + \lambda^2 - 3 \\ &= \lambda^2 - 5\lambda - 7 \\ |B - \lambda I| &= \begin{vmatrix} 1 - \lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} = (1 - \lambda)(1 - \lambda) \end{aligned}$$

Since  $A$  and  $B$  do not have the same characteristic polynomial, they are not similar.

### Exercise 3

Show that  $A$  and  $B$  are not similar matrices.

$$\begin{aligned} A &= \begin{bmatrix} 2 & 1 & 4 \\ 0 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \\ B &= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 2 & 3 & 4 \end{bmatrix} \\ |A - \lambda I| &= (2 - \lambda)(2 - \lambda)(4 - \lambda) \\ |B - \lambda I| &= (1 - \lambda)(4 - \lambda)(4 - \lambda) \end{aligned}$$

Since  $A$  and  $B$  do not have the same characteristic polynomial, they are not similar.

### Exercise 5

A diagonalization of the matrix  $A$  is given in form  $P^{-1}AP = D$ . List the eigenvalues of  $A$  and bases from the corresponding eigenspaces.

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\lambda = 4 \quad \lambda = 3$$

$$E_4 = \text{span} \left( \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$$

$$E_3 = \text{span} \left( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$$

### Exercise 7

A diagonalization of the matrix  $A$  is given in form  $P^{-1}AP = D$ . List the eigenvalues of  $A$  and bases from the corresponding eigenspaces.

$$\begin{bmatrix} \frac{1}{8} & \frac{1}{8} & \frac{1}{8} \\ -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{5}{8} & -\frac{3}{8} & -\frac{3}{8} \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 2 & 0 & 2 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\lambda = 6 \quad \lambda = -2 \quad \lambda = -2$$

$$E_6 = \text{span} \left( \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right)$$

$$E_{-2} = \text{span} \left( \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right)$$

### Exercise 9

Determine whether  $A$  is diagonalizable and, if so, find an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $P^{-1}AP = D$ .

$$A = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -3 - \lambda & 4 \\ -1 & 1 - \lambda \end{vmatrix} = (-3 - \lambda)(1 - \lambda) + 4$$

$$= -3 + 2\lambda + \lambda^2 + 4 = \lambda^2 + 2\lambda + 1$$

$$= (\lambda + 1)^2 = 0$$

$A$  is not diagonalizable since there cannot be two linearly independent eigenvectors.

### Exercise 11

Determine whether  $A$  is diagonalizable and, if so, find an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $P^{-1}AP = D$ .

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & 0 & 1 \\ 0 & 1 - \lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = -\lambda^3 + 2\lambda^2 + \lambda - 2$$
$$= -\lambda^2(\lambda - 2) + 1(\lambda - 2)$$
$$= (1 - \lambda)(1 + \lambda)(\lambda - 2) = 0$$
$$\lambda = 1 \quad \lambda = -1 \quad \lambda = 2$$
$$[A - 1I|0] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$
$$E_1 = \text{span} \left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$$
$$[A + 1I|0] = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$
$$E_{-1} = \text{span} \left( \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right)$$
$$[A - 2I|0] = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$
$$E_2 = \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

The algebraic multiplicities of the eigenvectors are equal to their geometric multiplicities, so the matrix is diagonalizable. The following matrices  $P$  and  $D$  satisfy  $P^{-1}AP = D$ .

$$P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 2 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

### Exercise 13

Determine whether  $A$  is diagonalizable and, if so, find an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $P^{-1}AP = D$ .

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$
$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & 2 & 1 \\ -1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \lambda^2(\lambda - 1) = 0$$
$$\lambda = 0 \quad \lambda = 1$$
$$[A - 0I|0] = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$
$$E_0 = \text{span} \left( \begin{pmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \end{pmatrix} \right)$$

Since the geometric multiplicity is not equal to the algebraic multiplicity for  $\lambda = 0$ , this matrix is not diagonalizable.

### Exercise 15

Determine whether  $A$  is diagonalizable and, if so, find an invertible matrix  $P$  and a diagonal matrix  $D$  such that  $P^{-1}AP = D$ .

$$A = \begin{bmatrix} 2 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$
$$\lambda = 2 \quad \lambda = 2$$
$$[A - 2I|0] = \begin{bmatrix} 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$x_3 = x_4 = 0$$
$$E_2 = \text{span} \left( \begin{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{pmatrix} \right)$$



$$[A + 2I|0] = \begin{bmatrix} 4 & 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = -x_4$$

$$x_2 = 0$$

$$\begin{aligned} E_{-2} &= \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ -x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \\ &= \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right) \end{aligned}$$

Since the algebraic multiplicities of the eigenvalues is equal to their respective geometric multiplicities, the matrix is diagonalizable. The following matrices  $P$  and  $D$  satisfy  $P^{-1}AP = D$ .

$$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

### Exercise 17

Use the method of Example 4.29 to compute the indicated power of the matrix.

$$\begin{aligned} A &= \begin{bmatrix} -1 & 6 \\ 1 & 0 \end{bmatrix} \\ |A - \lambda I| &= \begin{vmatrix} -1 - \lambda & 6 \\ 1 & -\lambda \end{vmatrix} \\ &= -\lambda(-1 - \lambda) - 6 = \lambda^2 + \lambda - 6 \\ &= (\lambda + 3)(\lambda - 2) = 0 \\ \lambda &= -3 \quad \lambda = 2 \\ [A + 3I|0] &= \begin{bmatrix} 2 & 6 & 0 \\ 1 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
E_{-3} &= \text{span} \left( \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right) \\
[A - 2I|0] &= \begin{bmatrix} -3 & 6 & 0 \\ 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
E_2 &= \text{span} \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \\
D &= \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix} \\
P &= \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \\
P^{-1} &= -1 \begin{bmatrix} -1 & -3 \\ -1 & 2 \end{bmatrix} \\
P^{-1}AP &= D \\
P^{-1}A^{10}P &= D^{10} \\
A^{10} &= PD^{10}P^{-1} \\
&= \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 \\ 0 & (-3)^{10} \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix}
\end{aligned}$$

### Exercise 19

Use the method of Example 4.29 to compute the indicated power of the matrix.

$$\begin{aligned}
A &= \begin{bmatrix} 0 & 3 \\ 1 & 2 \end{bmatrix} \\
|A - \lambda I| &= \begin{vmatrix} -\lambda & 3 \\ 1 & 2 - \lambda \end{vmatrix} = -\lambda(2 - \lambda) - 3 \\
&= \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0 \\
\lambda &= 3 \quad \lambda = -2 \\
[A - 3I|0] &= \begin{bmatrix} -3 & 3 & 0 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
E_3 &= \text{span} \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \\
[A + 1I|0] &= \begin{bmatrix} 1 & 3 & 0 \\ 1 & 3 & 0 \end{bmatrix} \\
E_{-2} &= \text{span} \left( \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right)
\end{aligned}$$

$$\begin{aligned}
D &= \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \\
P &= \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix} \\
P^{-1} &= \frac{1}{2} \begin{bmatrix} -1 & -3 \\ 1 & 1 \end{bmatrix} \\
P^{-1}AP &= D \\
P^{-1}A^kP &= D^k \\
A^k &= PD^kP^{-1} \\
&= \begin{bmatrix} 1 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & (-2)^k \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}
\end{aligned}$$

### Exercise 21

Use the method of Example 4.29 to compute the indicated power of the matrix.

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
|A - \lambda I| &= (1 - \lambda)(-1 - \lambda)(-1 - \lambda) = 0 \\
\lambda &= -1 \quad \lambda = 1 \\
[A - 1I|0] &= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
E_1 &= \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) \\
[A + 1I|0] &= \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
E_{-1} &= \begin{bmatrix} x_1 \\ x_2 \\ -x_2 - 2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \\
&= \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)
\end{aligned}$$

$$\begin{aligned}
D &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \\
P &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix} \\
P^{-1} &= \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix} \\
A^{2015} &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1^{2015} & 0 & 0 \\ 0 & (-1)^{2015} & 0 \\ 0 & 0 & (-1)^{2015} \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 \end{bmatrix}
\end{aligned}$$

### Exercise 23

Use the method of Example 4.29 to compute the indicated power of the matrix.

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 1 & 0 \\ 2 & -2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \\
|A - \lambda I| &= \begin{vmatrix} 1 - \lambda & 1 & 0 \\ 2 & -2 - \lambda & 2 \\ 0 & 1 & 1 - \lambda \end{vmatrix} = (1 - \lambda)((-2 - \lambda)(1 - \lambda) - 2) - 2(1 - \lambda) \\
&= (1 - \lambda)(\lambda^2 + \lambda - 4) - 2 + 2\lambda \\
&= \lambda^2 + \lambda - 4 - \lambda^3 - \lambda^2 + 4\lambda - 2 + 2\lambda \\
&= -\lambda^3 + 7\lambda - 6 = 0 \\
\lambda &= -3 \quad \lambda = 1 \quad \lambda = 2 \\
[A + 3I|0] &= \begin{bmatrix} 4 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 0 & 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
E_{-3} &= \text{span} \left( \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right) \\
[A - 1I|0] &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & -3 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
E_1 &= \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) \\
[A - 2I|0] &= \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\
E_2 &= \text{span} \left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)
\end{aligned}$$

$$\begin{aligned}
D &= \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
P &= \begin{bmatrix} 1 & 1 & 1 \\ 4 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \\
P^{-1} &= \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix} \\
A^k &= \begin{bmatrix} 1 & 1 & 1 \\ 4 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} (-3)^k & 0 & 0 \\ 0 & 1^k & 0 \\ 0 & 0 & 2^k \end{bmatrix} \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & -\frac{1}{6} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}
\end{aligned}$$

### Exercise 25

Find all real values of  $k$  for which  $A$  is diagonalizable.

$$\begin{aligned}
A &= \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \\
|A - \lambda I| &= (1 - \lambda)^2 = 0 \\
\lambda &= 1 \\
[A - 1I|0] &= \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
E_1 &= \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)
\end{aligned}$$

The matrix is not diagonalizable as long as  $k \neq 0$  since the geometric multiplicity of  $\lambda = 1$  will be less than its algebraic multiplicity. This matrix is only diagonalizable for  $k = 0$ .

### Exercise 27

Find all real values of  $k$  for which  $A$  is diagonalizable.

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
|A - \lambda I| &= (1 - \lambda)^3 = 0 \\
\lambda &= 1 \\
[A - 1I|0] &= \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\
E_1 &= \text{span} \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)
\end{aligned}$$

By the same logic as Exercise 25, this matrix is only diagonalizable if  $k = 0$ .

### Exercise 29

Find all real values of  $k$  for which  $A$  is diagonalizable.

$$\begin{aligned} A &= \begin{bmatrix} 1 & 1 & k \\ 1 & 1 & k \\ 1 & 1 & k \end{bmatrix} \\ |A - \lambda I| &= \begin{vmatrix} 1 - \lambda & 1 & k \\ 1 & 1 - \lambda & k \\ 1 & 1 & k - \lambda \end{vmatrix} = \\ &= (k - k + k\lambda) - (k - k\lambda - k) + (k - \lambda)((1 - \lambda)^2 - 1) \\ &= k\lambda + k\lambda + (k - \lambda)(\lambda^2 - 2\lambda) \\ &= 2k\lambda + k\lambda^2 - 2k\lambda - \lambda^3 + 2\lambda^2 \\ &= -\lambda^3 + (2 + k)\lambda^2 \\ &= \lambda^2(-\lambda + 2 + k) = 0 \\ \lambda = 0 \quad \lambda = k + 2 \\ [A - 0I|0] &= \begin{bmatrix} 1 & 1 & k & 0 \\ 1 & 1 & k & 0 \\ 1 & 1 & k & 0 \end{bmatrix} \\ E_0 &= \text{span} \left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} k \\ 0 \\ -1 \end{bmatrix} \right) \\ [A - (k + 2)I|0] &= \begin{bmatrix} -k - 1 & 1 & k & 0 \\ 1 & -k - 1 & k & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 2 & 0 \\ 1 & -k - 1 & k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

$A$  is diagonalizable for all real values of  $k$ .

If you have any questions, comments, or concerns, please contact me at [alvin@omgimanerd.tech](mailto:alvin@omgimanerd.tech)