Linear Algebra: Homework 7

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Section 3.5

Exercise 1

Let S be the collection of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ that satisfy the given property. In each case, either prove that S forms a subspace of $\mathbb{R}^2$ or give a counterexample to show that it does not.

$$x = 0$$

$$S = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = span\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Since $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in \mathbb{R}^2$, S is a subspace of $\mathbb{R}^2$.

Exercise 3

Let S be the collection of vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$ that satisfy the given property. In each case, either prove that S forms a subspace of $\mathbb{R}^2$ or give a counterexample to show that it does not.

$$y = 2x$$

$$S = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 2x \end{bmatrix} = x \begin{bmatrix} 1 \\ 2 \end{bmatrix} = span\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

Since $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$, S is a subspace of $\mathbb{R}^2$.

Exercise 5

Let S be the collection of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ that satisfy the given property. In each case, either prove that S forms a subspace of $\mathbb{R}^2$ or give a counterexample to show that it does not.

$$x = y = z$$

$$S = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = span\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

Since $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \mathbb{R}^3$, S is a subspace of $\mathbb{R}^3$.

Exercise 7

Let S be the collection of vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ in $\mathbb{R}^3$ that satisfy the given property. In each case, either prove that S forms a subspace of $\mathbb{R}^2$ or give a counterexample to show that it does not.

$$x - y + z = 1$$

$$0 - 0 + 0 \neq 1$$

Since the zero vector is not in S, S is not a subspace of $\mathbb{R}^3$.

Exercise 9

Prove that every line through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = span\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}\right)$$

Every line through the origin can be described as the span of a vector $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \in \mathbb{R}^3$. Therefore, every line through the origin is a subspace of $\mathbb{R}^3$.

Exercise 10

Suppose S consists of all points in $\mathbb{R}^2$ that are on the x-axis or the y-axis (or both). Is S a subspace of $\mathbb{R}^2$? Why or why not?

$$S = \left\{\begin{bmatrix} x \\ 0 \end{bmatrix} \mid x \in \mathbb{R}\right\} \cup \left\{\begin{bmatrix} 0 \\ y \end{bmatrix} \mid y \in \mathbb{R}\right\}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in S$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \in S$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin S$$

The subspace S is not closed under addition.

Exercise 15

If A is the matrix in Exercise 11, is $\vec{v} = \begin{bmatrix} -1 \\ 3 \\ -1 \end{bmatrix}$ in $null(A)$?

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore \vec{v} \notin null(A)$$

Exercise 16

If A is the matrix in Exercise 12, is $\vec{v} = \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix}$ in $null(A)$?

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \\
A\vec{v} &= \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 1 \\ 1 & -1 & -4 \end{bmatrix} \begin{bmatrix} 7 \\ -1 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\therefore \vec{v} &\in null(A)
\end{aligned}$$

Exercise 17

Give bases for $row(A), col(A),$ and $null(A)$.

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \\
\text{basis for } row(A) &= \left\{ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \right\} \\
\text{basis for } col(A) &= \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \\
A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= \vec{0} \\
1x_1 - 1x_3 &= 0 \\
1x_2 + 2x_3 &= 0 \\
null(A) &= \begin{bmatrix} x_1 \\ -2x_1 \\ x_1 \end{bmatrix} \\
&= x_1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\
\text{basis for } null(A) &= \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}
\end{aligned}$$

Exercise 19

Give bases for $row(A), col(A),$ and $null(A)$.

$$
\begin{aligned}
A &= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\text{basis for } row(A) &= \left\{ \begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right\} \\
\text{basis for } col(A) &= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\} \\
A\vec{x} &= 0 \\
x_1 + x_3 &= 0 \\
x_2 - x_3 &= 0 \\
x_4 &= 0 \\
null(A) &= \begin{bmatrix} x_1 \\ -x_1 \\ -x_1 \\ 0 \end{bmatrix} \\
\text{basis for } null(A) &= \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} \right\}
\end{aligned}
$$

Exercise 27

Find a basis for the span of the given vectors.

$$
\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}
$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\text{basis} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Exercise 29

Find a basis for the span of the given vectors.

$$\begin{bmatrix} 2 & -3 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \quad \begin{bmatrix} 4 & -4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -3 & -1 \\ 1 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\text{basis} = \{e_1, e_2, e_3\}$$

Exercise 35

Give the rank and nullity of the matrix in Exercise 17.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$
$$rank(A) = 2$$
$$nullity(A) = 3 - rank(A) = 1$$

Exercise 37

Give the rank and nullity of the matrix in Exercise 19.

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$rank(A) = 3$$
$$nullity(A) = 4 - rank(A) = 1$$

Exercise 39

If A is a 3×5 matrix, explain why the columns of A must be linearly dependent.

$$\begin{aligned} rank(A) &\leq min(m, n) \\ A\vec{x} &= \vec{0} \\ 5 &= rank(A) + nullity(A) \\ rank(A) &\leq 3 \end{aligned}$$

The nullity of A must be at least 2, therefore $A\vec{x} = \vec{0}$ has a non-trivial solution. Thus, the columns of A are linearly dependent. Since there are 5 columns and only 3 can be linearly independent, they must be linearly dependent.

Exercise 40

If A is a 4×2 matrix, explain why the rows of A must be linearly dependent.
There are four rows in total, but only two rows can be linearly independent in a 4×2 matrix, therefore the rows must be linearly dependent.

Exercise 41

If A is a 3×5 matrix, what are the possible values of $nullity(A)$?

$$\begin{aligned} rank(A) &\leq 3 \\ n &= rank(A) + nullity(A) \\ 5 &= rank(A) + nullity(A) \\ 5 - nullity(A) &\leq 3 \\ -nullity(A) &\leq -2 \\ nullity(A) &\geq 2 \\ 2 &\leq nullity(A) \leq 5 \end{aligned}$$

Exercise 42

If A is a 4×2 matrix, what are the possible values of $nullity(A)$?

$$\begin{aligned} rank(A) &\leq 2 \\ n &= rank(A) + nullity(A) \\ 2 &= rank(A) + nullity(A) \\ 2 - nullity(A) &\leq 2 \\ -nullity(A) &\leq 0 \\ nullity(A) &\geq 0 \\ 0 &\leq nullity(A) \leq 2 \end{aligned}$$

Exercise 43

Find all possible values of $rank(A)$ as a varies.

$$\begin{aligned}
A &= \begin{bmatrix} 1 & 2 & a \\ -2 & 4a & 2 \\ a & -2 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 1 & -\frac{2}{a} & \frac{1}{a} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -\frac{2}{a}-2 & \frac{1}{a}-a \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & a \\ 0 & 4a+4 & 2+2a \\ 0 & -2-2a & 1-a^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & a \\ 0 & 2a+2 & 1+a \\ 0 & 2a+2 & a^2-1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 2 & a \\ 0 & 2a+2 & 1+a \\ 0 & 0 & a^2-a-2 \end{bmatrix} \\
2a+2 &= 0 \\
a &= -1 \\
&\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\
a^2-a-2 &= 0 \\
(a-2)(a+1) &= 0 \\
a &= 1, -2 \\
&\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 0 \end{bmatrix} \\
&\rightarrow \begin{bmatrix} 1 & 2 & a \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \\
rank(A) &= \begin{cases} 1 & \text{if } a = 1 \\ 2 & \text{if } a = -1 \vee a = 2 \\ 3 & \text{otherwise} \end{cases}
\end{aligned}$$

Exercise 44

Find all possible values of $rank(A)$ as a varies.

$$A = \begin{bmatrix} a & 2 & -1 \\ 3 & 3 & -2 \\ -2 & -1 & a \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \frac{2}{a} & -\frac{1}{a} \\ 1 & 2 & -2+a \\ -2 & -1 & a \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \frac{2}{a} & -\frac{1}{a} \\ 1 & 2 & -2+a \\ 0 & 3 & a+2(-2+a) \end{bmatrix}$$
$$= \begin{bmatrix} 1 & \frac{2}{a} & -\frac{1}{a} \\ 0 & 2-\frac{2}{a} & -2+a+\frac{1}{a} \\ 0 & 3 & 3a-4 \end{bmatrix}$$
$$= \begin{bmatrix} a & 2 & -1 \\ 0 & 2a-2 & -2a+a^2+1 \\ 0 & 3 & 3a-4 \end{bmatrix}$$
$$a = 1$$
$$\rightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & -1 \end{bmatrix}$$
$$rank(A) = \begin{cases} 2 & \text{if } a = 1 \\ 3 & \text{otherwise} \end{cases}$$

Exercise 45

Do $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form a basis for $\mathbb{R}^3$?

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the vectors are linearly independent and form the standard basis vectors, they form a basis for $\mathbb{R}^3$.

Exercise 47

Do $\begin{bmatrix}1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\\1\end{bmatrix}$ for a basis for $\mathbb{R}^4$?

$$\begin{bmatrix}1&1&1&0\\1&1&0&1\\1&0&1&1\\0&1&1&1\end{bmatrix} = \begin{bmatrix}1&1&1&0\\0&0&-1&1\\0&-1&0&1\\0&1&1&1\end{bmatrix}$$
$$= \begin{bmatrix}1&1&1&0\\0&0&-1&1\\0&0&1&2\\0&1&1&1\end{bmatrix}$$
$$= \begin{bmatrix}1&1&1&0\\0&0&0&3\\0&0&1&2\\0&1&1&1\end{bmatrix}$$
$$= \begin{bmatrix}1&0&0&0\\0&0&0&1\\0&0&1&0\\0&1&0&0\end{bmatrix}$$
$$= \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

Since the vectors form the standard basis vectors, they form a basis for $\mathbb{R}^4$.

Exercise 51

Show that w is in $span(B)$ and find the coordinate vector $[w]_B$.

$$B = \left\{ \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix} \right\} \quad \vec{w} = \begin{bmatrix}1\\6\\2\end{bmatrix}$$

$$c_1 \begin{bmatrix}1\\2\\0\end{bmatrix} + c_2 \begin{bmatrix}1\\0\\-1\end{bmatrix} = \begin{bmatrix}1\\6\\2\end{bmatrix}$$
$$\begin{bmatrix}1&1&1\\2&0&6\\0&-1&2\end{bmatrix} = \begin{bmatrix}1&1&1\\0&-2&4\\0&-1&2\end{bmatrix}$$
$$= \begin{bmatrix}1&0&3\\0&1&-2\\0&0&0\end{bmatrix}$$

$$[w]_B = \begin{bmatrix}3\\-2\end{bmatrix}$$

Exercise 52

Show that w is in $span(B)$ and find the coordinate vector $[w]_B$.

$$B = \left\{ \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} \right\} \quad \vec{w} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$$

$$\begin{aligned}
c_1 \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 1 \\ 6 \end{bmatrix} &= \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \\
\begin{bmatrix} 3 & 5 & 1 \\ 1 & 1 & 3 \\ 4 & 6 & 4 \end{bmatrix} &= \begin{bmatrix} 0 & 2 & -8 \\ 1 & 1 & 3 \\ 2 & 3 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & -4 \\ 1 & 1 & 3 \\ 0 & 1 & -4 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix} \\
[w]_B &= \begin{bmatrix} 7 \\ -4 \end{bmatrix}
\end{aligned}$$

Exercise 57

If A is $m \times n$, prove that every vector in $null(A)$ is orthogonal to every vector in $row(A)$.

$$\begin{aligned}
A\vec{x} &= \vec{0} \quad \forall \vec{x} \in \mathbb{R}^m \\
\vec{y} &= \sum_{i=1}^{m} c_i a_i \quad \forall \vec{u} \in row(A) \\
&= \begin{bmatrix} col_1(A) & \dots & col_m(A) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \\
&= A^T \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \\
x^T y &= x^T A^T c \\
x^T y &= (Ax)^T c \\
x \cdot y &= 0c = 0
\end{aligned}$$

Exercise 58

If A and B are $n \times n$ of rank n, prove that AB has rank n.

$$\begin{aligned}
nullity(A) &= 0 \\
nullity(B) &= 0 \\
B\vec{x} &= 0 \\
A(B\vec{x}) &= 0 \\
(AB)\vec{x} &= 0 \\
nullity(AB) &= 0 \\
rank(AB) &= n - nullity(AB) = n
\end{aligned}$$

Exercise 59a

Prove that $rank(AB) \leq rank(B)$.

$$\begin{aligned}
n &= rank(AB) + nullity(AB) \\
&= rank(B) + nullity(B) \\
nullity(B) &= dim(null(B)) \\
&\leq dim(null(AB)) \\
&\leq nullity(AB) \\
rank(AB) + nullity(AB) &= rank(B) + nullity(B) \\
\therefore rank(AB) &\leq rank(B)
\end{aligned}$$

Exercise 60a

Prove that $rank(AB) \leq rank(A)$.

$$\begin{aligned}
AB &= A \begin{bmatrix} col_1(B) & \dots & col_n(B) \end{bmatrix} \\
col(AB) &\subseteq col(B) \\
dim(col(AB)) &\leq dim(col(B)) \\
rank(AB) &\leq rank(B)
\end{aligned}$$

Exercise 61

Prove that if U is invertible, then $rank(UA) = rank(A)$.

$$\begin{aligned}
A &= IA \\
A &= U^{-1}UA \\
rank(U) &= n \\
\therefore rank(A) &= n = rank(U)
\end{aligned}$$

Prove that if V is invertible, then $rank(AV) = rank(A)$.

$$\begin{aligned}
A &= AI \\
A &= AV^{-1}V \\
rank(U) &= n \\
\therefore rank(A) &= n = rank(V)
\end{aligned}$$

Exercise 62

Prove that an $m \times n$ matrix A has rank 1 if and only if A can be written as the outer product uv^T of a vector $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$.
Suppose:

$$\begin{aligned}
A &= \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} w \\
u &= \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \\
w &= v^T \\
A &= uv^T \\
rank(A) &= rank(uv^T) \\
&\leq rank(u) \\
&\leq 1 \\
rank(A) &= 1
\end{aligned}$$

Exercise 63

Prove that an $m \times n$ matrix A has rank r, prove that A can be written as the sum of r matrices, each of which has rank 1.

$$\begin{aligned}
rank(A) &= r \\
&= dim(row(A)) \\
A &= \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} \\
&= \begin{bmatrix} c_{11}v_1 + c_{12}v_2 + \cdots + c_{1r}v_r \\ c_{21}v_1 + c_{22}v_2 + \cdots + c_{2r}v_r \\ \vdots \\ c_{m1}v_1 + v_{m2}v_2 + \cdots + c_{mr}v_r \end{bmatrix} \\
&= \begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix} v_1 + \cdots + \begin{bmatrix} c_{1r} \\ c_{2r} \\ \vdots \\ c_{mr} \end{bmatrix} v_r
\end{aligned}$$

Because $\begin{bmatrix} c_{11} \\ c_{21} \\ \vdots \\ c_{m1} \end{bmatrix}$, has rank 1, each of the r matrices has rank 1.

Exercise 64

Prove that, for $m \times n$ matrices A and B, $rank(A+B) \leq rank(A) + rank(B)$.

$$row_i(A+B) = row_i(A) + row_i(B)$$

The rows of $A+B$ can be expressed as linear combinations of the respective rows of A and B.

Exercise 65

Let A be an $n \times n$ matrix such that $A^2 = 0$. Prove that $rank(A) \leq \frac{n}{2}$.

$$\begin{aligned}
A^2 &= A\begin{bmatrix} col_1(A) & \dots & col_n(A) \end{bmatrix} \\
&= 0 \\
A\vec{x} &= 0 \\
col(A) &\subseteq null(A) \\
rank(A) + nullity(A) &= n \\
rank(A) + rank(A) &\leq n \\
2rank(A) &\leq n \\
rank(A) &\leq \frac{n}{2}
\end{aligned}$$

Exercise 66

Let A be a skew-symmetric $n \times n$ matrix.

- Prove that $x^T Ax = 0$ for all $x \in \mathbb{R}^n$.

$$\begin{aligned}
x^T Ax &= (x^T Ax)^T) \\
&= (Ax)^T (x^T)^T \\
&= x^T A^T x \\
&= x^T(-A)x \\
x^T Ax &= -x^T Ax \\
\therefore x^T Ax &= 0
\end{aligned}$$

- Prove that $I + A$ is invertible.
  If this is true $(I + A)x = 0$ has only the trivial solution.

$$\begin{aligned}
(I + A)x &= 0 \\
x + Ax &= 0 \\
x^T x + x^T Ax &= 0(x^T) \\
x^T x + 0 &= 0 \\
\therefore x &= 0
\end{aligned}$$

Section 3.6

Exercise 1

Let $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ be the matrix transformation corresponding to $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$. Find $T_A(\vec{u})$ and $T_A(\vec{v})$, where $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

$$T_A(\vec{u}) = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$
$$T_A(\vec{v}) = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

Exercise 3

Prove that the given transformation is a linear transformation, using the definition.

$$\begin{aligned}
T \begin{bmatrix} x \\ y \end{bmatrix} &= \begin{bmatrix} x + y \\ x - y \end{bmatrix} \\
T(u + v) &= T\left(\begin{bmatrix} u_x + v_x \\ u_y + v_y \end{bmatrix} \right) \\
&= \begin{bmatrix} u_x + v_x + u_y + v_y \\ u_x + v_x - u_y - v_y \end{bmatrix} \\
&= \begin{bmatrix} (u_x + u_y) + (v_x + v_y) \\ (u_x - u_y) + (v_x - v_y) \end{bmatrix} \\
&= T(u) + T(v) \\
T(cu) &= \begin{bmatrix} cu_x + cu_y \\ cu_x - cu_y \end{bmatrix} \\
&= c \begin{bmatrix} u_x + u_y \\ u_x - u_y \end{bmatrix} \\
&= cT(u)
\end{aligned}$$

Exercise 5

Prove that the given transformation is a linear transformation, using the definition.

$$T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x-y+z\\2x+y-3z\end{bmatrix}$$

$$\begin{aligned}
T(u+v) &= T\left(\begin{bmatrix}u_x+v_x\\u_y+v_y\end{bmatrix}\right)\\
&= \begin{bmatrix}(u_x+v_x)-(u_y+v_y)+(u_z+v_z)\\2(u_x+v_x)+(u_y+v_y)-3(u_z+v_z)\end{bmatrix}\\
&= \begin{bmatrix}(u_x-u_y+u_z)+(v_x-v_y+v_z)\\(2u_x+u_y-3u_z)+(2v_x+v_y-3v_z)\end{bmatrix}\\
&= T(u)+T(v)\\
T(cu) &= \begin{bmatrix}cu_x-cu_y+cu_z\\c2u_x+cu_y-c3u_z\end{bmatrix}\\
&= c\begin{bmatrix}u_x-u_y+u_z\\2u_x+u_y-3u_z\end{bmatrix}\\
&= cT(u)
\end{aligned}$$

Exercise 7

Give a counterexample to show that the given transformation is not a linear transformation.

$$T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}y\\x^2\end{bmatrix}$$

$$\begin{aligned}
T\left(2\begin{bmatrix}3\\3\end{bmatrix}\right) &= \begin{bmatrix}6\\36\end{bmatrix}\\
&\neq 2T\left(\begin{bmatrix}3\\3\end{bmatrix}\right)\\
&\neq \begin{bmatrix}6\\18\end{bmatrix}
\end{aligned}$$

Exercise 9

Give a counterexample to show that the given transformation is not a linear transformation.

$$T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}xy\\x+y\end{bmatrix}$$

$$\begin{aligned}
T\left(2\begin{bmatrix}3\\3\end{bmatrix}\right) &= \begin{bmatrix}36\\12\end{bmatrix}\\
&\neq 2T\left(\begin{bmatrix}3\\3\end{bmatrix}\right)\\
&\neq \begin{bmatrix}18\\12\end{bmatrix}
\end{aligned}$$

Exercise 11

Find the standard matrix of the linear transformation.

$$T\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x+y\\x-y\end{bmatrix}$$
$$= \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

Exercise 13

Find the standard matrix of the linear transformation.

$$T\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x-y+z\\2x+y-3z\end{bmatrix}$$
$$= \begin{bmatrix}1 & -1 & 1\\2 & 1 & -3\end{bmatrix}$$

Exercise 40

Use matrices to prove the given statements about the transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. If R_θ denotes a rotation (about the origin) through the angle θ, then $R_\alpha \circ R_\beta = R_\alpha + R_\beta$.

$$\begin{aligned}
R_\alpha &= \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\\
R_\beta &= \begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}\\
R_\alpha \circ R_\beta &= \begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{bmatrix}\\
&= \begin{bmatrix}\cos\alpha\cos\beta - \sin\alpha\sin\beta & -\cos\alpha\sin\beta - \sin\alpha\cos\beta\\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & -\sin\alpha\sin\beta + \cos\alpha\cos\beta\end{bmatrix}\\
&= \begin{bmatrix}\cos\alpha\cos\beta - \sin\alpha\sin\beta & -(\cos\alpha\sin\beta + \sin\alpha\cos\beta)\\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta\end{bmatrix}\\
&= \begin{bmatrix}\cos(\alpha+\beta) & -\sin(\alpha+\beta)\\ \sin(\alpha+\beta) & \cos(\alpha+\beta)\end{bmatrix}\\
&= R_{\alpha+\beta}
\end{aligned}$$

Exercise 42

(a) If P is a projection, then $P \circ P = P$.

$$\begin{aligned}
P &= \begin{bmatrix}\frac{d_x^2}{d_x^2+d_y^2} & \frac{d_xd_y}{d_x^2+d_y^2}\\ \frac{d_xd_y}{d_x^2+d_y^2} & \frac{d_y^2}{d_x^2+d_y^2}\end{bmatrix}\\
P \circ P &= \begin{bmatrix}\frac{d_x^4+d_x^2d_y^2}{(d_x^2+d_y^2)^2} & \frac{d_x^3d_y+d_xd_y^3}{(d_x^2+d_y^2)^2}\\ \frac{d_x^3d_y+d_xd_y^3}{(d_x^2+d_y^2)^2} & \frac{d_y^4+d_x^2d_y^2}{(d_1^2+d_2^2)^2}\end{bmatrix}\\
&= P
\end{aligned}$$

Exercise 44

Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Prove that T maps a straight line to a straight line or a point.

$$\begin{aligned} l &= \vec{x} + t\vec{d} \\ T(\vec{x} + t\vec{d}) &= T(\vec{x}) + T(t\vec{d}) \\ &= T(\vec{x}) + tT(\vec{d}) \end{aligned}$$

When $t = 0$, the result is a point, otherwise, the resulting mapping is a line.

Exercise 52

Prove that $P_l(c\vec{v}) = cP_l(\vec{v})$ for an scalar c.

$$\begin{aligned} P_l(c\vec{v}) &= \left(\frac{\vec{d} \cdot (c\vec{v})}{\vec{d} \cdot \vec{d}} \vec{d} \right) \\ &= \left(\frac{c(\vec{d} \cdot \vec{v})}{\vec{d} \cdot \vec{d}} \vec{d} \right) \\ &= c \left(\frac{\vec{d} \cdot \vec{v}}{\vec{d} \cdot \vec{d}} \vec{d} \right) \\ &= cP_l(\vec{v}) \end{aligned}$$

Exercise 53

Prove that $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if:

$$T(c_1\vec{v_1} + c_2\vec{v_2}) = c_1T(\vec{v_1}) + c_2T(\vec{v_2})$$

$$\begin{aligned} T(c_1\vec{v_1} + c_2\vec{v_2}) &= T(c_1\vec{v_1}) + T(c_2\vec{v_2}) \\ &= c_1T(\vec{v_1}) + c_2T(\vec{v_2}) \end{aligned}$$

Exercise 54

Prove that (as noted at the beginning of this section) the range of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the column space of its matrix $[T]$.

$$\begin{aligned} [T] &= \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} \\ \vec{u} &= \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \\ T(\vec{u}) &= u_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix} + \dots + u_n \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \end{aligned}$$

The range of T is a linear combination of the columns, so it is a subset of the column space. The converse is also true, the column space is a subset of the range. Therefore, the two sets are equal.

If you have any questions, comments, or concerns, please contact me at alvin@omgimanerd.tech